PMATH451 — Measure and Integration

Class notes for Fall 2019

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Preface

Assignment problems is introduced in class as we go, so we have the special environment homework for these in this note.

I included a special chapter in the appendix (see Appendix A) that records and provides insights into what drove the direction(s) of certain proofs. This is an attempt to resolve the problem of proofs being overly obscure with its motivations. Contents presented in this appendix are typically like rough work, and so are typically much longer than the presented proof.

I also made an appendix for some of the common themes and tricks (see Appendix B) that are seen repeatedly in this topic I think it is invaluable that they are noted down, because the ideas that these commonalities carry forward.

1 Lecture 1 Sep 04th, 2019

1.1 Motivation for the Study of Measures

Recall Riemann integration.

Definition (Riemann Integration)

*Let $f : [a, b] \to \mathbb{R}$ be a **bounded** function. We call*

$$P = \{a = x_0 < x_1 < \ldots < x_n = b\} \subseteq [a, b]$$

*a **partition** of $[a, b]$, and*

$$\Delta x_i = x_i - x_{i-1}$$

*as the **length of the i^{th} interval** for $i = 1, \ldots, n$.*

Let

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

*be the **supremum of f on the i^{th} interval**, and*

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$$

*be the **infimum of f on the i^{th} interval**. We define the **Riemann upper sum** as*

$$U(f, P) = \sum_i M_i \Delta x_i,$$

*and the **Riemann lower sum** as*

$$L(f, P) = \sum_i m_i \Delta x_i.$$

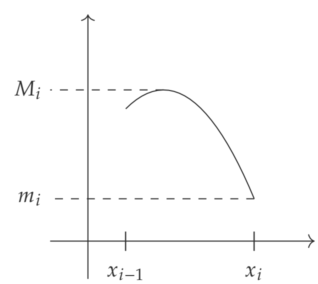

Figure 1.1: Idea of Riemann integration

*We define the **Riemann upper integral** as*

$$\overline{\int_a^b} f\,dx = \inf_P U(f, P)$$

*and the **Riemann lower integral** as*

$$\underline{\int_a^b} f\,dx = \sup L(f, P).$$

*We say that f is **Riemann integrable** if*

$$\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx,$$

and we write the integral of f as

$$\int_a^b f\,dx = \overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx.$$

As hyped up as one does earlier in university about Riemann integration, there are functions that are not Riemann integrable!

Example 1.1.1

Consider a function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}.$$

Then

$$\overline{\int_a^b} f\,dx = 1 \text{ and } \underline{\int_a^b} f\,dx = 0.$$

Thus f is not Riemann integrable.

Note 1.1.1 (Shortcomings of the Riemann integral)

1. *We cannot characterize functions that are Riemann integrable, i.e. we do not have a list of characteristics that we can check against to see if a function is Riemann integrable.*

This remained an open problem in the earlier 1920s.

2. *The Riemann integral behaves badly when it comes to pointwise limits of functions. The next example shall illustrate this.*

3. *The Riemann integral is awkward when f is unbounded. In particular, we used to hack our way around by looking at whether the Riemann integral converges to some value the function approaches the unbounded point, and then "conclude" that the integral is the limit of that convergence.*

4. *Recall that the* ***Fundamental Theorem of Calculus*** *states that*

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x).$$

We know that this works for Riemann integrals. By the first shortcoming, the problem here is that we do not fully know what are the functions that the Fundamental Theorem is true for.

5. *In PMATH450, we saw that Fourier developed the Fourier series, which is an extremely useful tool in solving* ***Differential Equations*** *using sines and cosines. However, the convergence of the Fourier series remains largely unexplained by Fourier, and we have but developed some roundabout ways of showing some convergence.*

6. *Consider the set R if Riemann integrable functions on the interval $[a, b]$. The set R has a natural metric:*

$$d(f, g) = \int_a^b \left| f - g \right| dx.$$

However, the metric space (R, d) is ***not complete****. This means many of our favorite results in PMATH351 are not usable!*

7. *There are many functions that seem like they should have an integral, but turned out that they did not under Riemann integration.*

Example 1.1.2 (Pointwise Limits of Riemann Integrable Functions is not necessarily Riemann Integrable)

Let $\mathbb{Q} = \{x_n\}_{n \in \mathbb{N}}$. Then consider a sequence of functions

$$f_n(x) = \begin{cases} 1 & x \in \{x_1, \ldots, x_n\} \\ 0 & x \notin \{x_1, \ldots, x_n\} \end{cases}.$$

It is rather clear that

$$\overline{\int_a^b} f \, dx = \underline{\int_a^b} f \, dx = 0.$$

However, the pointwise limit of the f_n's, and that is

$$\lim_{n \to \infty} f_n(x) = f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases},$$

is, as mentioned in the last example, not Riemann integrable.

To address the shortcomings of the Riemann integral, Henri Lebesgue developed the **Lebesgue integral**, of which we have seen in PMATH450.

Instead of dividing the x-axis, Lebesgue decided to divide the y-axis first.

If the range of a function f is $[c, d]$, where c, d can be infinite, then we partition the interval such that

$$P = \{c = y_0 < y_1 < \ldots < y_n = d\},$$

and we define

$$E_i = \{x : f(x) \in [y_{i-1}, y_i]\}.$$

Then if A_i is the area of the "rectangle" for the i^{th} interval of $[c, d]$, we have

$$y_{i-1} \cdot \ell(E_i) \leq A_i \leq y_i \cdot \ell(E_i),$$

where $\ell(E_i)$ is the **Lebesgue measure** of the set E_i. Then if we let $\int_a^b f$ denote the Lebesgue integral of f, we would expect

$$\sum_{i=1}^{n} y_{i-1} \cdot \ell(E_i) \leq \int_a^b f \leq \sum_{i=1}^{n} y_i \cdot \ell(E_i).$$

However, to truly understand what this means, we need to understand what the Lebesgue measure is.

Furthermore, recall that in PMATH450, we saw that not all sets, in $\mathbb{R}$ for example, are measurable, and for 'good' reasons, there always exists non-measurable sets.

1.2 *Algebras and σ-Algebra of Sets*

For this course, we shall use the convention that

- the 'ambient' space X is always non-empty;
- $\mathcal{P}(X)$, the power set of X, has non-trivial elements; and
- we denote $A^C = \{x \in X : x \notin A\}$ for $A \subseteq X$.

Definition 1 (Algebra of Sets)

*Given X, a non-empty collection of subsets of X, i.e. $\emptyset \neq \mathcal{A} \subseteq \mathcal{P}(X)$, is called an **algebra of sets** of X provided that:*

1. $A_1, \ldots, A_n \in \mathcal{A} \implies \bigcup_{i=1}^n A_i \in \mathcal{A}$; *and*
2. $A \in \mathcal{A} \implies A^C \in \mathcal{A}$.

Proposition 1 (Properties of Algebra of Sets)

If $\mathcal{A}$ is an algebra of sets of X, then

3. $\emptyset, X \in \mathcal{A}$;
4. $A, B \in \mathcal{A} \implies A \setminus B = \{x \in X \mid x \in A \wedge x \notin B\} \in \mathcal{A}$; *and*
5. $A_1, \ldots, A_n \in \mathcal{A} \implies \bigcap_{i=1}^n A_i \in \mathcal{A}$.

Proof

3. $\mathcal{A} \neq \emptyset \implies \exists A \in \mathcal{A} \implies A^C \in \mathcal{A} \implies A \cup A^C = X \in \mathcal{A} \implies \emptyset = X^C \in \mathcal{A}$.
4. $A, B \in \mathcal{A} \implies A^C \in \mathcal{A} \implies A^C \cup B \in \mathcal{A} \implies A \setminus B = (A^C \cup B)^C \in \mathcal{A}$.
5. (**De Morgan's Law**) Notice that $(A_1 \cap A_2 \cap \ldots \cap A_n)^C = A_1^C \cup A_2^C \cup \ldots A_n^C \in \mathcal{A}$ since $A_i^C \in \mathcal{A}$. Thus the complement
$$A_1 \cap A_2 \cap \ldots \cap A_n \in \mathcal{A}.$$
□

Definition 2 (σ-Algebra of Sets)

*Given X and $\emptyset \neq \mathcal{A} \subseteq \mathcal{P}(X)$, we say that $\mathcal{A}$ is a **σ-algebra of sets** of X if it is an algebra of sets and*

$$\forall A_n \in \mathcal{A}, n \in \mathbb{N}, \quad \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}.$$

Example 1.2.1

1. $\mathcal{P}(X)$ is a σ-algebra.

2. Consider X as an infinite set. We say that a set A is **cofinite** if A^C is finite. Let

$$\mathcal{A} := \{A \in \mathcal{P}(X) \mid A \text{ is finite or cofinite }\}.$$

Then $\mathcal{A}$ is an algebra of sets:

- finite union of finite sets remains finite;
- finite union of finite and cofinite sets remains cofinite; and
- complement of finite sets are the cofinite sets and vice versa.

However, $\mathcal{A}$ is **not** a σ-algebra: consider $A_n = \{2^n\} \subseteq X = \mathbb{N}$, which we then realize that

$$\bigcup_{n \in \mathbb{N}} A_n = \text{ set of all even numbers },$$

but the set of all even numbers is clearly not finite, and its complement, which is the set of all odd numbers, is not finite.

3. Consider X as an uncountable set. We say that a set A is **co-countable** if A^C is countable. [1] The set

$$\mathcal{A} := \{A \subseteq X \mid A \text{ is countable or co-countable }\}$$

is a σ-algebra:

- countable union of countable sets is countable;

[1] Recall that a set A is said to be countable if there is a one-to-one correspondence between elements of A and the natural numbers.

- countable union of countable and co-countable sets is co-countable; and
- complement of countable sets are co-countable and vice versa.

2 Lecture 2 Sep 06th 2019

2.1 Algebra and σ-algebra of Sets (Continued)

We've seen some examples of σ-algebras. Let's now look at some other important properties of σ-algebras.

Proposition 2 (Closure of σ-algebras under Countable Intersection)

Let X be a set, $\mathcal{A}$ a σ-algebra on X. If $A_n \in \mathcal{A}$ for each $n \in \mathbb{N}$, then $\bigcap_n A_n \in \mathcal{A}$.

This follows rather similarly to Proposition 1 where we used **De Morgan's Law**.

Proof

We observe that

$$\begin{aligned} A_n \in \mathcal{A} &\implies A_n^C \in \mathcal{A} \\ &\implies \bigcup_n A_n^C \in \mathcal{A} \\ &\implies \bigcap_n A_n = \left(\bigcup_n A_n^C \right)^C \in \mathcal{A}. \end{aligned}$$

□

Let $\mathcal{A}_\alpha \subseteq \mathcal{P}(X)$, where α is from some index set. We denote

$$\bigcap_\alpha \mathcal{A}_\alpha = \{A \subseteq X : A \in \mathcal{A}_\alpha, \ \forall \alpha\}.$$

Proposition 3 (Existence of the 'Smallest' σ-algebra on a Set)

Let X be a set and $\{\mathcal{A}_\alpha\}_\alpha$ as a collection of σ-algebras on X. Then $\bigcap_\alpha A_\alpha$ is a σ-algebra.

Proof

$$\begin{aligned} A \in \bigcap_\alpha \mathcal{A}_\alpha &\implies \forall \alpha, \, A \in \mathcal{A}_\alpha \\ &\implies \forall \alpha, \, A^C \in \mathcal{A}_\alpha \\ &\implies A^C \in \bigcap_\alpha \mathcal{A}_\alpha \end{aligned}$$

and

$$\begin{aligned} \forall n \in \mathbb{N}, \, A_n \in \bigcap_\alpha \mathcal{A}_\alpha &\implies \forall n \in \mathbb{N}, \forall \alpha, \, A_n \in \mathcal{A}_\alpha \\ &\implies \forall \alpha, \bigcup_n A_n \in \mathcal{A}_\alpha \\ &\implies \bigcup_n A_n \in \bigcap_\alpha \mathcal{A}_\alpha. \end{aligned}$$ □

Due to the above proposition, the following definition is well-defined.

Definition 3 (Generator of a σ-algebra)

Let X be a set, and $\xi \subseteq \mathcal{P}(X)$ has some non-trivial set(s). Consider all σ-algebras $\mathcal{A}_\alpha$ with the property that $\xi \subseteq \mathcal{A}_\alpha$. Then we say that $\bigcap_\alpha \mathcal{A}_\alpha$

is the σ-algebra generated by ξ, and we denote this generated σ-algebra as

$$\mathfrak{M}(\xi) = \bigcap_{\alpha} \mathcal{A}_{\alpha}.$$

Remark 2.1.1

1. *It is clear from the definition that if $\mathcal{A}$ is a σ-algebra on X and $\xi \subseteq \mathcal{A}$, then $\mathfrak{M}(\xi) \subseteq \mathcal{A}$.*
2. *We often say that $\mathfrak{M}(\xi)$ is the "smallest σ-algebra containing ξ".*

The following is an example of such a σ-algebra.

Definition 4 (Borel σ-algebra)

Let X be a metric space (or topological space). The σ-algebra generated by the ***open subsets*** *of X is called the* ***Borel σ-algebra****, of which we denote by $\mathfrak{B}(X)$.*

Remark 2.1.2 (Some sets in $\mathfrak{B}(X)$)

Given an arbitrary metric space (or topological space) X. It is often hard to firmly grasp what kind of sets are in the Borel σ-algebra $\mathfrak{B}(X)$. The following are some examples that are in $\mathfrak{B}(X)$.

1. *Let $\{\mathcal{O}_n\}_{n \in \mathbb{N}}$ denote a countable collection of open sets. By Proposition 2, $\bigcap_n \mathcal{O}_n \in \mathfrak{B}(X)$. We call these countable union of open sets as G_δ sets.*
2. *Let $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ denote a countable collection of closed sets. By Proposition 2, $\bigcup_n \mathcal{F}_n \in \mathfrak{B}(X)$. We call these countable intersection of closed sets as F_σ sets.*
3. *Let $\{H_n\}$ be a countable collection of G_δ sets. Then $\bigcup_n H_n \in \mathfrak{B}(X)$. These are called the $G_{\delta\sigma}$ sets.*
4. *Let $\{K_n\}$ be a countable collection of F_σ sets. Then $\bigcap_n K_n \in \mathfrak{B}(X)$. These are called the $F_{\sigma\delta}$ sets.*

We can continue constructing the $G_{\delta\sigma\dots}$ and $F_{\sigma\delta\dots}$ similarly, and all these sets belong to the Borel σ-algebra $\mathfrak{B}(X)$.

Proposition 4 (Other Formulations of the Borel σ-algebra (aka Proposition 1.2))

The following collection of sets are all equal:

1. $\mathfrak{B}_1 = \mathfrak{B}(\mathbb{R})$;
2. $\mathfrak{B}_2 =$ *σ-algebra generated by open intervals (e.g. (a, b));*
3. $\mathfrak{B}_3 =$ *σ-algebra generated by closed intervals (e.g. $[a, b]$);*
4. $\mathfrak{B}_4 =$ *σ-algebra generated by half-open intervals (e.g. $(a, b]$);*
5. $\mathfrak{B}_5 =$ *σ-algebra generated by $(-\infty, a)$ and (b, ∞); and*
6. $\mathfrak{B}_6 =$ *σ-algebra generated by $(-\infty, a]$ and $[b, \infty)$.*

As commented before, it is often hard knowing that is in a Borel σ-algebra, and what is not, despite knowing what its generator is. However, when talking about containments, this is a fairly straightforward discussion thanks to its closure under countable unions and Proposition 2. We simply need to talk about the generators.

Proof

$\mathfrak{B}_2 \subseteq \mathfrak{B}_1$ Given an arbitrary generator (a, b) in $\mathfrak{B}_2$, we know that (a, b) is an open set, and clearly $(a, b) \subseteq \mathbb{R}$. Thus $(a, b) \in \mathfrak{B}_1$, so $\mathfrak{B}_2 \subseteq \mathfrak{B}_1$.

$\mathfrak{B}_3 \subseteq \mathfrak{B}_2$ Given an arbitrary generator $[a, b]$ of $\mathfrak{B}_2$, we have

$$[a, b] = \bigcap_n \left(a - \frac{1}{n}, b + \frac{1}{n} \right) \in \mathfrak{B}_2.$$

Thus $\mathfrak{B}_3 \subseteq \mathfrak{B}_2$.

$\mathfrak{B}_4 \subseteq \mathfrak{B}_3$ Given an arbitrary generator $(a, b]$ of $\mathfrak{B}_4$,

$$(a, b] = \bigcup_n \left[a + \frac{1}{n}, b \right] \in \mathfrak{B}_3.$$

Thus $\mathfrak{B}_4 \subseteq \mathfrak{B}_3$.

$\mathfrak{B}_5 \subseteq \mathfrak{B}_4$ Given an arbitrary generator $(-\infty, a)$ for $\mathfrak{B}_5$,

$$(-\infty, a) = \bigcup_n \left(-\infty, a - \frac{1}{n} \right) \in \mathfrak{B}_4.$$

On the other hand, for (b, ∞) in $\mathfrak{B}_5$,

$$(b, \infty) = \bigcup_n (b, n) \in \mathfrak{B}_4.$$

$\mathfrak{B}_6 \subseteq \mathfrak{B}_5$ We have that

$$(-\infty, a] = \bigcap_n \left(-\infty, a + \frac{1}{n} \right) \in \mathfrak{B}_5$$

and

$$[b, \infty) = \bigcap_n \left(b - \frac{1}{n}, \infty \right) \in \mathfrak{B}_5.$$

$\mathfrak{B}_1 \subseteq \mathfrak{B}_6$ Let $c < d \in \mathbb{R}$. Notice that

$$(-\infty, d] \cap [c, \infty) = [c, d] \in \mathfrak{B}_6.$$

Furthermore,

$$(c, d) = \bigcup_n \left[c + \frac{1}{n}, d - \frac{1}{n} \right] \in \mathfrak{B}_6.$$

Recall that given an open set $\mathcal{O} \subseteq \mathbb{R}$, we have

$$\mathcal{O} = \bigcup \{ (c, d) \subseteq \mathcal{O} : c, d \in \mathbb{Q} \},$$

which shows that $\mathcal{O}$ is a countable union of open sets (with rational endpoints). It follows that $\mathcal{O} \in \mathfrak{B}_6$ and so $\mathfrak{B}_1 \subseteq \mathfrak{B}_6$. $\square$

Exercise 2.1.1

Show that $\mathfrak{B}(\mathbb{R}^2)$ *is generated by open rectangles* $(a, b) \times (c, d)$.

Definition 5 (Infinitely Often)

Given $E_n \subseteq X$ for $n \in \mathbb{N}$, we say that $x \in E_n$ ***infinitely often (i.o.)*** *if*

$$\{n : x \in E_n\}$$

is an ***infinite set****. We typically let*

$$A := \{x \in X : x \in E_n \text{ i.o. }\}$$

be the set of x's that are in the E_n's infinitely often.

Definition 6 (Almost always)

Given $E_n \subseteq X$ for $n \in \mathbb{N}$, we say that $x \in E_n$ ***almost always (a.a.)*** *if*

$$\{n : x \notin E_n\}$$

is a ***finite set****. We typically let*

$$B := \{x \in X : x \in E_n \text{ a.a. }\}$$

be the set of x's that are in the E_n's almost always.

Homework (Homework 1)

Let X be a set, $\mathcal{A}$ a σ-algebra on X, and $E_n \in \mathcal{A}$ for $n \in \mathbb{N}$. Prove that

$$A := \{x \in X : x \in E_n \text{ i.o. }\}$$

and

$$B := \{x \in X : x \in E_n \text{ a.a. }\}$$

are both in $\mathcal{A}$.

Definition 7 (Characteristic Function)

Let $E \subseteq X$. We call the function

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

the ***characteristic function*** *of E.*

Homework (Homework 2 – A review on limsup and liminf)

Let $E_n \subseteq X$ for $n \in \mathbb{N}$, and

$$\begin{aligned} A &:= \{x \in X : x \in E_n \text{ i.o. }\} \\ B &:= \{x \in X : x \in E_n \text{ a.a. }\}. \end{aligned}$$

Show that

$$\begin{aligned} \chi_A(x) &= \limsup_n \chi_{E_n}(x) \\ \chi_B(x) &= \liminf_n \chi_{E_n}(x). \end{aligned}$$

Remark 2.1.3

Due to the above result, some people write

$$\begin{aligned} A &= \limsup E_n \\ B &= \liminf E_n. \end{aligned}$$

2.2 Measures

Definition 8 (Measure)

Let X be a set and $\mathcal{A}$ a σ-algebra of subsets of X. A function $\mu : \mathcal{A} \to [0, \infty]$ is called a ***measure*** *on $\mathcal{A}$ provided that:*

1. $\mu(\emptyset) = 0$; *and*

2. *if* $E_n \in \mathcal{A}$ *for each* $n \in \mathbb{N}$*, and* $\{E_n\}$ *is disjoint, we have*

$$\mu\left(\bigcup_n E_n\right) = \sum_n \mu(E_n).$$

3 Lecture 3 Sep 9th 2019

3.1 Measures (Continued)

Definition 9 (Measure Space)

*Let X be a set, $\mathfrak{M}$ a σ-algebra of subsets of X and $\mu : \mathfrak{M} \to [0, \infty]$. We call the 3-tuple $(X, \mathfrak{M}, \mu)$ a **measure space**.*

Remark 3.1.1

*If $\mu(X) = 1$, we also call $(X, \mathfrak{M}, \mu)$ a **probability space**, and μ is called a **probability measure**.*

Example 3.1.1

1. (**Counting Measure**) Let X be a set and $\mathfrak{M} = \mathcal{P}(X)$. For $E \in \mathfrak{M}$, define

$$\mu(E) = \begin{cases} |E| & E \text{ is finite} \\ \infty & \text{otherwise} \end{cases}.$$

We verify that μ is indeed a measure:

(a) We have that $\mu(\emptyset) = |\emptyset| = 0$.

(b) Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathfrak{M}$ be a pairwise disjoint set. Notice that if any of the sets are infinite, say E_{N_0} is infinite, then

$$\mu(E_{N_0}) = \infty = \left|E_{N_0}\right|.$$

Since $\bigcup_{n=1}^{\infty} E_n$ is infinite in this case, we have

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \infty = \left|\bigcup_{n=1}^{\infty} E_n\right|.$$

On the other hand, if all the sets are finite, then since the E_n's are disjoint, we have

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \left|\bigcup_{n=1}^{\infty} E_n\right| = \sum_{n=1}^{\infty} |E_n| = \sum_{n=1}^{\infty} \mu(E_n).$$

We call μ a counting measure.

2. Let X be an uncountable set. Recall that in Example 1.2.1, we showed that

$$\mathfrak{M} := \{A \subseteq X \mid A \text{ is countable or co-countable }\}$$

is a σ-algebra. There are many measures that we can define on this σ-algebra. For instance,

$$\nu(E) = \begin{cases} 0 & E \text{ is countable} \\ 1 & E \text{ is uncountable} \end{cases},$$

and

$$\delta(E) = \begin{cases} 0 & E \text{ is countable} \\ \infty & E \text{ is uncountable} \end{cases}.$$

Verifying that both ν and δ are indeed measures shall be left to the reader as a straightforward exercise.

3. Let's make a non-example. Let X be an infinite set, and $\mathfrak{M} = \mathcal{P}(X)$. Define

$$\mu(E) = \begin{cases} 0 & E \text{ is finite} \\ \infty & E \text{ is infinite} \end{cases}.$$

Consider $X = \mathbb{N}$ and a sequence of sets with singletons,

$$E_n = \{2n+1\}, \quad \text{for } n \in \mathbb{N}.$$

Clearly,

$$\bigcup_{n=1}^{\infty} E_n = \text{ set of all odd numbers },$$

and clearly

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \infty.$$

However, notice that

$$\mu(E_n) = 0 \text{ for each } n \in \mathbb{N}.$$

Since each of the E_n's are pairwise disjoint, we should have

$$\infty = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) = 0,$$

which is impossible. Thus μ is **not** a measure.

Remark 3.1.2 (Finite additivity)

Given a finite set of pairwise disjoint sets $\{E_n\}_{n=1}^{N} \subseteq \mathfrak{M}$ for some σ-algebra $\mathfrak{M}$ of some set X. By the definition of a σ-algebra, we may set $E_n = \emptyset$ for $n > N$. Then

$$\mu\left(\bigcup_{n=1}^{N} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{N} \mu(E_n).$$

*We call this the **finite additivity** of a measure.*

Definition 10 (Finitivity, σ-finitivity, and Semi-finitivity of a Measure)

Let $(X, \mathfrak{M}, \mu)$ be a measure space.

1. *We say that μ is **finite** if $\mu(E) < \infty$ for every $E \in \mathfrak{M}$.*
2. *If $X = \bigcup_{n=1}^{\infty} X_n$ with $X_n \in \mathfrak{M}$, we say that μ is **σ-finite** if*

$$\mu(X_n) < \infty \text{ for every } n \in \mathbb{N}.$$

3. *We say that μ is **semi-finite** if for every $E \in \mathfrak{M}$ with $\mu(E) = \infty$,*

$\exists F \subseteq E \in \mathfrak{M}$ *such that*

$$0 < \mu(F) < \infty.$$

Exercise 3.1.1

1. *Show that the counting measure is finite iff the ambient space X is a finite set.*
2. *Show that δ in Example 3.1.1 is neither finite, σ-finite, nor semi-finite.*

Theorem 5 (Properties of a Measure)

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Then

1. *(Monotonicity) If $E \subseteq F$ and $E, F \in \mathfrak{M}$, then $\mu(E) \leq \mu(F)$.*
2. *(Subadditivity) If $\{E_n\}_{n=1}^{\infty} \subseteq \mathfrak{M}$, then*
$$\mu\left(\bigcup_n E_n\right) \leq \sum_n \mu(E_n).$$
3. *(Continuity from below) If $\{E_n\}_{n=1}^{\infty} \subseteq \mathfrak{M}$ is an increasing sequence of sets, i.e.*
$$E_1 \subseteq E_2 \subseteq \ldots \subseteq E_n \subseteq \ldots,$$
then
$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} \mu(E_n).$$
4. *(Continuity from above) If $\{E_n\}_{n=1}^{\infty} \subseteq \mathfrak{M}$ is a decreasing sequence of sets, i.e.*
$$E_1 \supseteq E_2 \supseteq \ldots \supseteq E_n \supseteq \ldots,$$
and $\exists n_0 \in \mathbb{N}$ such that $\mu(E_{n_0}) < \infty$, then
$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} \mu(E_n).$$

Remark 3.1.3 (A comment on the condition for the 4^{th} statement)

It may seem that the extra condition of a finite measure seem extravagant. However, it is necessary, as demonstrated below.

Consider $X = \mathbb{N}$, with μ as the counting measure. Then, consider the sequence of sets

$$\begin{aligned} E_1 &= \{1, 2, 3, \ldots\}, \\ E_2 &= \{2, 3, 4, \ldots\}, \\ E_3 &= \{3, 4, 5, \ldots\}, \\ &\vdots \\ E_n &= \{n, n+1, n+2, \ldots\}, \\ &\vdots \end{aligned}$$

Then $\bigcap_{n=1}^{\infty} E_n = \emptyset$, which then $\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = 0$. However,

$$\mu(E_n) = \infty \text{ for each } n \in \mathbb{N}.$$

Homework (Homework 3)

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathfrak{M}$, and

$$A := \{x \in X \mid x \in E_n \text{ i.o. }\}.$$

Prove that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$ implies that $\mu(A) = 0$.

4 Lecture 4 Sep 11th 2019

4.1 Measures (Continued 2)

We shall now prove Theorem 5.

Proof

1. Notice that
$$F = (F \cap E) \cup (F \setminus E),$$
and $F \cap E$ and $F \setminus E$ are disjoint. Thus
$$\mu(F) = \mu(F \cap E) + \mu(F \setminus E) = \mu(E) + \mu(F \setminus E).$$
Since $\mu(F \setminus E) \geq 0$, we have
$$\mu(F) \geq \mu(E).$$

2. Consider a sequence of sets defined as such: [1]
$$\begin{aligned} F_1 &= E_1 \\ F_2 &= E_2 \setminus E_1 \\ &\vdots \\ F_n &= E_n \setminus \bigcup_{j=1}^{n-1} E_j. \end{aligned}$$
First, note that $F_n \subseteq E_n$ for each $n \in \mathbb{N}$. So by the last part, we have
$$\mu(F_n) \leq \mu(E_n) \text{ for each } n \in \mathbb{N}.$$

[1] ★ This is a common technique in measure theory. We will see this repeatedly so in this course.

Secondly,

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n.$$

Also, $\{F_n\}_{n=1}^{\infty}$ is a pairwise disjoint collection of sets. It follows that

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

3. Consider a sequence of sets defined as such:

$$\begin{aligned}
F_1 &= E_1 \\
F_2 &= E_2 \setminus E_1 \\
F_3 &= E_3 \setminus E_2 \\
&\vdots \\
F_n &= E_n \setminus E_{n-1}.
\end{aligned}$$

We see that

- $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$;
- $\bigcup_{n=1}^{N} F_n = \bigcup_{n=1}^{N} E_n = E_N$; and
- $\{F_n\}_n$ is a collection pairwise disjoint sets.

Thus we have

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} E_n\right) &= \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n) \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \mu(F_n) = \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{N} F_n\right) \\
&= \lim_{N\to\infty} \mu(E_N).
\end{aligned}$$

4. First, it is important that we notice that

$$\bigcap_{n=1}^{\infty} E_n = \bigcap_{n=m}^{\infty} E_n$$

for any $m \in \mathbb{N}$, since $\{E_n\}_n$ is a decreasing sequence of sets.

Suppose $n_0 \in \mathbb{N}$ is such that $\mu(E_{n_0}) < \infty$. Consider a sequence of sets defined as follows: for $n_0 \leq j \in \mathbb{N}$, we let $F_j = E_{n_0} \setminus E_j$. Then we have

$$\emptyset = F_{n_0} \subseteq F_{n_0+1} \subseteq \ldots \subseteq F_{n_0+k} \subseteq \ldots,$$

i.e. $\{F_n\}_{n=n_0}^{\infty}$ is an increasing sequence of sets. By the last part, we have

$$\begin{aligned} \mu\left(\bigcup_{n=n_0}^{\infty} F_n\right) &= \lim_{n\to\infty} \mu(F_{n_0+n}) = \lim_{n\to\infty} \mu(E_{n_0} \setminus E_{n_0+n}) \\ &= \mu(E_{n_0}) - \lim_{n\to\infty} \mu(E_{n_0+n}) \\ &= \mu(E_{n_0}) - \lim_{n\to\infty} \mu(E_n). \end{aligned}$$

Furthermore, we observe that

$$\bigcup_{n=1}^{\infty} F_n = E_{n_0} \setminus \bigcap_{n=n_0}^{\infty} E_n.$$

Thus

$$\begin{aligned} \mu\left(\bigcup_{n=n_0}^{\infty} F_n\right) &= \mu\left(E_{n_0} \setminus \bigcap_{n=n_0}^{\infty} E_n\right) = \mu(E_{n_0}) - \mu\left(\bigcap_{n=n_0}^{\infty} E_n\right) \\ &= \mu(E_{n_0}) - \mu\left(\bigcap_{n=1}^{\infty} E_n\right). \end{aligned}$$

It follows that indeed

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n). \qquad \square$$

Exercise 4.1.1

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Show that

1. *μ is finite iff $\mu(X) < \infty$.*
2. *μ is σ-finite implies that μ is semi-finite.*

Solution

1. This is rather simple.

   ($\implies$) μ is finite implies that each $E \in \mathfrak{M}$ has a finite measure. In particular, $X \in \mathfrak{M}$, and so $\mu(X) < \infty$.

($\Longleftarrow$) $\forall E \in \mathfrak{M}$, $E \subseteq X$, thus by the first item in Theorem 5, we have $\mu(E) \leq \mu(X) < \infty$. Thus μ is finite.

2. μ being σ-finite means that if $X = \bigcup_{n=1}^{\infty} X_n$ where $X_n \in \mathfrak{M}$, then $\mu(X_n) < \infty$ for each n. Let $E \in \mathfrak{M}$ such that $\mu(E) = \infty$. If we take

$$E_n = X_n \cap E,$$

then $\mu(E_n) < \infty$ for each $n \in \mathbb{N}$. Then, taking a union of any finite number of these E_n's will give us a subset of E with a finite measure. Hence, μ is indeed semi-finite.

Definition 11 (Null Set of a Measure)

Let $(X, \mathfrak{M}, \mu)$ be a measure space. The set

$$\mathcal{N} := \{N \in \mathfrak{M} : \mu(N) = 0\}$$

is called the ***μ-null set****, or the* ***null set*** *of the measure μ.*

Remark 4.1.1

1. *If $N_j \in \mathcal{N}$, then $\bigcup_{n=1}^{\infty} N_j \in \mathcal{N}$.* [2]
2. *If $N \in \mathcal{N}$, and $E \in \mathfrak{M}$ and $E \subseteq N$, then $E \in \mathcal{N}$.*

   It is important to note there that the highlighted condition is required, since not all subsets of N are measurable.
3. *$\mathcal{N}$ is* ***not*** *a σ-algebra. If we picked an X such that $\mu(X) \neq 0$, then $\emptyset \in \mathcal{N}$ but $X \notin \mathcal{N}$.*

[2] Requires elab

Definition 12 (Complete Measure Space)

Let $(X, \mathfrak{M}, \mu)$ be a measure space. We say that the space is ***complete*** *if $N \in \mathcal{N}$ and $E \subseteq N$, then $E \in \mathfrak{M}$. In this case, we also say that μ is a* ***complete measure*** *on $\mathfrak{M}$.*

Remark 4.1.2

By the first item in Theorem 5, we have that if $\mu(E) = 0$, and so $E \in \mathcal{N}$ as well.

Theorem 6 (Extending the Measurable Sets)

Let $(X, \mathfrak{M}, \mu)$ be a measure space and

$$\mathcal{N} := \{N \in \mathfrak{M} \mid \mu(N) = 0\}.$$

Consider

$$\overline{\mathfrak{M}} := \{E \cup F \mid E \in \mathfrak{M},\ F \subseteq N \in \mathcal{N}\}.$$

Then $\overline{\mathfrak{M}}$ is a σ-algebra which contains $\mathfrak{M}$. Furthermore, if we define $\overline{\mu} : \overline{\mathfrak{M}} \to [0, \infty]$ as

$$\overline{\mu}(E \cup F) = \mu(E),$$

then $\overline{\mu}$ is a well-defined measure on $\overline{\mathfrak{M}}$.

Moreover, if $\nu : \overline{\mathfrak{M}} \to [0, \infty]$ is any measure such that $\nu(E) = \mu(E)$ for all $E \in \mathfrak{M}$, then $\nu = \overline{\mu}$.

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5 Lecture 5 Sep 13th 2019

5.1 Measures (Continued 3)

Proof (Extending the Measurable Sets)

$\overline{\mathfrak{M}}$ is a σ-algebra Since $\emptyset \in \mathfrak{M}$ and $\emptyset \subseteq N$ for any $N \in \mathcal{N}$, it is clear that $\emptyset \in \overline{\mathfrak{M}}$.

Now, for $E \cup F \in \overline{\mathfrak{M}}$, if we suppose $F \subseteq N \in \mathcal{N}$, then

$$(E \cup F)^C = (E \cup N)^C \cup (N \setminus E \cup F) \in \overline{\mathfrak{M}}$$

since $E \cup N \in \mathfrak{M}$ and $N \setminus (E \cup F) \in \mathcal{N}$.

Let $\{E_n \cup F_n\}_{n=1}^{\infty} \subseteq \overline{\mathfrak{M}}$. Then we observe that

$$\bigcup_{n=1}^{\infty}(E_n \cup F_n) = \underbrace{\bigcup_{n=1}^{\infty} E_n}_{\in \mathfrak{M}} \cup \underbrace{\bigcup_{n=1}^{\infty} F_n}_{\in \mathcal{N}} \in \overline{\mathfrak{M}}.$$

Well-definedness of $\overline{\mu}$ Let $E_1 \cup F_1 = E_2 \cup F_2 \in \overline{\mathfrak{M}}$. Suppose $F_1 \subseteq N_1, F_2 \subseteq N_2 \in \mathcal{N}$. WTS

$$\mu(E_1) = \overline{\mu}(E_1 \cup F_1) = \overline{\mu}(E_2 \cup F_2) = \mu(E_2)$$

Notice that

$$E_1 \subseteq E_1 \cup F_1 = E_2 \cup F_2 \subseteq E_2 \cup N_2,$$

and

$$E_2 \subseteq E_2 \cup F_2 = E_1 \cup F_1 \subseteq E_1 \cup N_1.$$

By Theorem 5, in particular, by subadditivity, we have that

$$\mu(E_1) \leq \mu(E_2 \cup N_2) \leq \mu(E_2) + 0 = \mu(E_2)$$

and

$$\mu(E_2) \leq \mu(E_1 \cup N_1) \leq \mu(E_1) + 0 = \mu(E_1).$$

It follows that $\mu(E_1) = \mu(E_2)$, as required.

$\overline{\mu}$ is a measure

1. Since $\emptyset \in \mathfrak{M}$ and $\emptyset \in \mathcal{N}$, $\overline{\mu}$ is defined for $\emptyset$, and

$$\overline{\mu}(\emptyset) = \mu(\emptyset) = 0.$$

2. Let $\{E_n \cup F_n\}_{n=1}^{\infty} \subseteq \overline{\mathfrak{M}}$ be a pairwise disjoint collection. We observe that

$$\begin{aligned}
\overline{\mu}\left(\bigcup_{n=1}^{\infty}(E_n \cup F_n)\right) &= \overline{\mu}\left(\bigcup_{n=1}^{\infty} E_n \cup \bigcup_{n=1}^{\infty} F_n\right) \\
&= \mu\left(\bigcup_{n=1}^{\infty} E_n\right) \\
&= \sum_{n=1}^{\infty} \mu(E_n),
\end{aligned}$$

and

$$\sum_{n=1}^{\infty} \overline{\mu}(E_n \cup F_n) = \sum_{n=1}^{\infty} \mu(E_n).$$

Hence

$$\overline{\mu}\left(\bigcup_{n=1}^{\infty}(E_n \cup F_n)\right) = \sum_{n=1}^{\infty} \overline{\mu}(E_n \cup F_n).$$

$\nu = \overline{\mu}$ Let $E \cup F \in \overline{\mathfrak{M}}$. Suppose $F \subseteq N \in \mathfrak{M}$ By monotonicity,

$$\overline{\mu}(E \cup F) = \mu(E) = \nu(E) \leq \nu(E \cup F).$$

By subadditivity,

$$\nu(E \cup F) \leq \nu(E) + \nu(F) \leq \mu(E) + \nu(N) \leq \overline{\mu}(E \cup F) + \mu(N) = \overline{\mu}(E \cup F) + 0.$$

Thus, indeed,

$$\nu(E \cup F) = \overline{\mu}(E \cup F).$$ □

5.2 The Outer Measure

In this section, we will show that one way we can construct a measure is by using an outer measure.

Definition 13 (Outer Measure)

Given a set X, a function

$$\mu^* : \mathcal{P}(X) \implies [0, \infty]$$

*is called an **outer measure** if*

1. $\mu^*(\emptyset) = 0$;
2. (*monotonicity*) *if $E \subseteq F$, then $\mu^*(E) \leq \mu^*(F)$; and*
3. (*countable subadditivity*) *if $\{A_n\}_n \subseteq \mathcal{P}(X)$, then*

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

Coming from PMATH450, we have seen an example of an outer measure.

Proposition 7 (Lebesgue's Outer Measure)

Given $E \subseteq \mathbb{R}$, consider

$$\mu^*(E) := \inf\left\{\sum_{n=1}^{\infty}(b_n - a_n) : E \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n)\right\}.$$

μ^* is Lebesgue's outer measure.

Proof

1. It is clear that $\mu^*(\emptyset) = \emptyset$, since we can pick all $(a_n, b_n) = \emptyset$.

2. Suppose $A \subseteq B \subseteq \mathbb{R}$. It is clear that any collection of intervals whose union contain B will contain A, but there are such collections for A that do not contain B. This means that

$$\mu^*(A) \leq \mu^*(B)$$

by the property of the infimum.

3. Let $E = \bigcup_{i=1}^{\infty} E_i$. WTS $\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$.

Now if $\mu^*(E_i) = \infty$ for any i, then the inequality is trivially true. Thus, wma $\mu^*(E_i) < \infty$ for all i.

[1] Let $\varepsilon > 0$. By the definition of the infimum, for each i, we can pick a countable sequence $\{(a_n^i, b_n^i)\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$ such that $E_1 \subseteq \bigcup_{n=1}^{\infty} (a_n^i, b_n^i)$ and

[1] This is also a common trick in measure theory.

$$\sum_{n=1}^{\infty} (b_n^i - a_n^i) \leq \mu^*(E_i) + \frac{\varepsilon}{2^i}.$$

Then

$$E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} (a_n^i, b_n^i).$$

And so it follows that

$$\begin{aligned} \mu^*(E) &\leq \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} (b_n^i - a_n^i) \\ &\leq \sum_{i=1}^{\infty} \mu^*(E_i) + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} \\ &= \sum_{i=1}^{\infty} \mu^*(E_i) + \varepsilon. \end{aligned}$$

Since ε was arbitrary, it follows that

$$\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i). \qquad \square$$

Exercise 5.2.1

Show that had we defined Lebesgue's outer measure with closed intervals, i.e.

$$\tilde{\mu}^*(E) := \inf \left\{ \sum_{n=1}^{\infty} (b_n - a_n) : E \subseteq \bigcup_{n=1}^{\infty} [a_n, b_n] \right\},$$

$\tilde{\mu}^*$ is still an outer measure.

In fact, we can do so for half-open intervals.

Example 5.2.1 (Lebesgue-Stieltjes Outer Measure)

Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function that is continuous from the right. Let

$$\mu^*(E) := \inf \left\{ \sum_{n=1}^{\infty} (F(b_n) - F(a_n)) : E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n] \right\}.$$

Then μ^* is an outer measure.

Remark 5.2.1

Again, we could have defined the above outer measure using open or closed intervals.

Example 5.2.2 (Lebesgue's Outer Measure on $\mathbb{R}^2$)

Let $E \subseteq \mathbb{R}^2$, and

$$\mu^*(E) := \inf \left\{ \sum_{n=1}^{\infty} A(R_n) : E \subseteq \bigcup_{n=1}^{\infty} R_n \right\},$$

where A is the 'area' function, and $R_n = (a_n, b_n) \times (c_n, d_n)$ are open rectangles. Then μ^* is an outer measure.

Remark 5.2.2

1. *Again, we can define the above outer measure using closed rectangles, or half-open rectangles.*
2. *We can continue defining an outer measure for $\mathbb{R}^3$ using cubes, for $\mathbb{R}^4$ using hypercubes, and so on.*

We want to now show that given an outer measure, we can always construct a measure. This is known as Carathéodory's Theorem.

This requires the following definition:

Definition 14 (μ^*-measurability)

*A set $A \subseteq X$ is said to be **μ^*-measurable** if $\forall E \subseteq X$,*

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C).$$

Remark 5.2.3

1. *By subadditivity, we always have*

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^C),$$

since $E = (E \cap A) \cup (E \cap A^C)$.

2. *Note that $E \cap A^C = E \setminus A$. In a sense, A is said to be μ^*-measurable if it can slice any subset of X such that we have additivity of the sliced parts. We may also say that A is a 'universal slicer'.*

Theorem 8 (Carathéodory's Theorem)

If μ^* is an outer measure on a set X, let

$$\mathfrak{M} := \{A \subseteq X : A \text{ is } \mu^*\text{-measureable}\}.$$

Then $\mathfrak{M}$ is a σ-algebra, and we set

$$\mu : \mathfrak{M} \to [0, \infty]$$

such that

$$\mu(A) = \mu^*(A).$$

Then μ is a complete measure on $\mathfrak{M}$.

6 Lecture 6 Sep 16th 2019

6.1 The Outer Measure (Continued)

Homework (Homework 4)

Let $\mathfrak{M}$ be an algebra of sets on X, and whenever $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathfrak{M}$ is a disjoint collection of sets, then $\bigcup_n A_n \in \mathfrak{M}$. Then $\mathfrak{M}$ is a σ-algebra.

Homework (Homework 5)

Recall that Lebesgue's Outer Measure on $\mathbb{R}$ is defined as

$$\mu^*(E) := \inf \left\{ \sum_{n=1}^{\infty} (b_n - a_n) : E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n) \right\}.$$

Prove that we can equivalently define

$$\mu^*(E) := \inf \left\{ \sum_{n=1}^{\infty} (b_n - a_n) : E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n] \right\}.$$

Similarly, Lebesgue's Outer Measure on $\mathbb{R}^2$ is defined as

$$\mu_2^*(E) = \inf \left\{ \sum_{n=1}^{\infty} (b_n - a_n)(d_n - c_n) : E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n) \times (c_n, d_n) \right\}.$$

Prove that we can equivalently define

$$\mu_2^*(E) = \inf \left\{ \sum_{n=1}^{\infty} (b_n - a_n)(d_n - c_n) : E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n] \times (c_n, d_n] \right\}.$$

Definition 15 (Metric Outer Measure)

Let (X, d) be a metric space, and $A, B \subseteq X$, and

$$d(A, B) = \inf \left\{ d(x, y) : x \in A,\ y \in B \right\}.$$

*An outer measure, μ^*, on X is called a **metric outer measure** if whenever $d(A, B) > 0$, then*

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$

Homework (Homework 6)

Prove that Lebesgue's Outer Measure on $\mathbb{R}$ is a metric outer measure.

Proof (Carathéodory's Theorem)

$\mathfrak{M}$ is a σ-algebra

$\emptyset \in \mathfrak{M}$ Given any $E \subseteq X$, we observe that

$$\begin{aligned} \mu^*(E \cap \emptyset) + \mu^*(E \cap \emptyset^C) &= \mu^*(\emptyset) + \mu^*(E \cap X) \\ &= 0 + \mu^*(E) = \mu^*(E). \end{aligned}$$

$A \in \mathfrak{M} \implies A^C \in \mathfrak{M}$ Observe that given any $E \subseteq X$,

$$\mu^*(E \cap A^C) + \mu^*(E \cap (A^C)^C) = \mu^*(E \cap A^C) + \mu^*(E \cap A) = \mu^*(E).$$

Thus $A^C \in \mathfrak{M}$.

[1] To show that $\mathfrak{M}$ is closed under countable unions, we break the work into several steps.

[1] For a deep dive, see Appendix A.1.

$A, B \in \mathfrak{M} \implies A \cup B \in \mathfrak{M}$ Since $A \in \mathfrak{M}$, we have

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C).$$

Since $B \in \mathfrak{M}$,

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^C) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^C) \\ &\quad + \mu^*(E \cap A^C \cap B) + \mu^*(E \cap A^C \cap B^C) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^C) \\ &\quad + \mu^*(E \cap A^C \cap B) + \mu^*(E \cap (A \cup B)^C)\end{aligned}$$

Notice that

$$E \cap (A \cup B) = [E \cap A \cap B] \,\dot\cup\, [E \cap A^C \cap B] \,\dot\cup\, [E \cap A \cap B^C].$$

Thus

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^C) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^C) \\ &\quad + \mu^*(E \cap A^C \cap B) + \mu^*(E \cap (A \cup B)^C) \\ &\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^C).\end{aligned}$$

Thus $A \cup B \in \mathfrak{M}$.

Consequently, by induction, we have that $\forall \{A_n\}_n \subseteq \mathfrak{M}$,

$$\bigcup_{n=1}^{N} A_n \in \mathfrak{M}$$

for all $N \in \mathbb{N}$.

Now $\mathfrak{M}$ is an algebra of sets. By Homework 4, it suffices for us to prove the following to show that $\mathfrak{M}$ is a σ-algebra of sets.

$\forall \{A_n\}_n \subseteq \mathfrak{M}$ disjoint, $\implies \dot\bigcup_n A_n \in \mathfrak{M}$ Let $B_N = \dot\bigcup_{n=1}^{N} A_n$. We first require the following lemma:

$\forall E \subseteq X,\ \mu^*(E \cap B_N) = \sum_{n=1}^{N} \mu^*(E \cap A_n)$ Notice that for any $n \in \mathbb{N}$, $A_n \in \mathfrak{M}$, and so

$$\begin{aligned}\mu^*(E \cap B_N) &= \mu^*(E \cap B_N \cap A_n) + \mu^*(E \cap B_N \cap A_n^C) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{N-1}).\end{aligned}$$

The desired result follows by induction. ⊣

Let $B = \dot\bigcup_{n=1}^{\infty} A_n$. Then

$$\mu^*(E \cap B) \leq \sum_{n=1}^{\infty} \mu^*(E \cap A_n)$$

by subadditivity.

Now $B_N \subseteq B$ for each $N \in \mathbb{N}$. This implies that $B_N^C \supseteq B^C$, and so by monotonicity,

$$\mu^*(E \cap B_N^C) \geq \mu^*(E \cap B^C).$$

Thus, for every $N \in \mathbb{N}$,

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap B_N) + \mu^*(E \cap B_N^C) \\ &\geq \sum_{n=1}^{N} \mu^*(E \cap A_n) + \mu^*(E \cap B^C).\end{aligned}$$

It follows that

$$\begin{aligned}\mu^*(E) &\geq \sum_{n=1}^{\infty} \mu^*(E \cap A_n) + \mu^*(E \cap B^C) \\ &\geq \mu^*(E \cap B) + \mu^*(E \cap B^C).\end{aligned}$$

With Homework 4, $\mathfrak{M}$ is a σ-algebra.

μ is a measure

- $\mu(\emptyset) = \mu^*(\emptyset) = 0$.
- Let $\{A_n\}_n \subseteq \mathfrak{M}$ be a disjoint collection of sets, and $B = \dot\bigcup_{n=1}^{\infty} A_n$. Then by a similar argument as the end of the last 'part',

$$\begin{aligned}\mu(B) &= \mu^*(B) \\ &\geq \sum_{n=1}^{\infty} \mu^*(B \cap A_n) + \mu^*(B \cap B^C) \\ &= \sum_{n=1}^{\infty} \mu^*(B \cap A_n) + 0 \\ &= \sum_{n=1}^{\infty} \mu^*(A_n) = \sum_{n=1}^{\infty} \mu(A_n).\end{aligned}$$

Thus

$$\mu(B) = \mu\left(\bigcupdot_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

μ is complete Let $A \in \mathcal{N}$ and $B \subseteq A$. By monotonicity, $\mu(B) = \mu^*(B) \leq \mu^*(A) = 0$. Then

$$\mu^*(E \cap B) + \mu^*(E \cap B^C) = 0 + \mu^*(E \cap B^C) \leq \mu^*(E)$$

by monotonicity. Thus $B \in \mathfrak{M}$. Thus μ is complete. □

We would like to make sure that

1. there are many sets that are measurable; and
2. the notion of a measure covers our notion of length.

We shall see this with the Metric Outer Measure, and that the measurable sets is at least the **Borel set**.

6.2 The Lebesgue-Stieltjes Outer Measure

The Lebesgue-Stieltjes outer measure is motivated by probability theory. The idea is that we consider the measure space $(\Omega, \mathfrak{M}, P)$, where Ω is the sample space set, $\mathfrak{M}$ is a σ-algebra on Ω, and P is the probability measure, i.e. $P(\Omega) = 1$.

We then define a **random variable**, which is a function $X : \Omega \to \mathbb{R}$. The **cumulative distribution function** (cdf) is defined as

$$F_X(t) := P(\{\omega : X(\omega) \leq t\}),$$

and it has these properties:

1. F_X is increasing; and
2. F_X is right-continuous.

Example 6.2.1

Let $\Omega = \{H, T\}$, and define the probability measure as

$$P(\{H\}) = \frac{1}{2} = P(\{T\}).$$

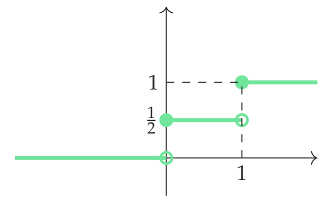

Figure 6.1: Simple example of a cdf

We can define
$$X(T) = 0 \text{ and } X(H) = 1.$$

Then
$$P(\{\omega : X(\omega) = 1\}) = P(\{H\}) = \frac{1}{2}$$
and
$$P(\{\omega : X(\omega) = 0\}) = P(\{T\}) = \frac{1}{2}.$$
In the context of probability, we often see the shorthand
$$P(X = t) = P(\{\omega : X(\omega) = t\}).$$

Definition 16 (Lebesgue-Stieltjes Outer Measure)

Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function that is continuous from the right. Let
$$\mu^*(E) := \inf \left\{ \sum_{n=1}^{\infty} (F(b_n) - F(a_n)) : E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n] \right\}.$$
Then μ^* is an outer measure.

Exercise 6.2.1

We mentioned that the above is indeed an outer measure in Example 5.2.1. Prove this.

7 Lecture 7 Sep 18th 2019

7.1 The Lebesgue-Stieltjes Outer Measure (Continued)

Theorem 9 (Carathéodory's Second Theorem)

Let (X, d) be a metric space, and μ^* a Metric Outer Measure. Then every Borel set is μ^*-measurable.

Proof

By Carathéodory's Theorem,

$$\mathfrak{M} = \{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\}$$

is a σ-algebra. Then our statement says that $\mathfrak{B}(X) \subseteq \mathfrak{M}$. Thus, it suffices for us to show that if $U \in \mathfrak{B}(X)$, i.e. if $U \subseteq X$ is open, then $U \in \mathfrak{M}$. In particular, WTS $\forall E \subseteq X$,

$$\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \cap U^C).$$

Again, by subadditivity,

$$E = (E \cap U) \cup (E \cap U^C) \implies \mu^{(}E) \leq \mu^*(E \cap U) + \mu^*(E \cap U^C).$$

Thus it suffices for us to show that

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^{(}E \cap U^C).$$

Now if $\mu^*(E) = \infty$, this is trivially true. WMA $\mu^*(E) < \infty$.

[1] Consider $\{A_k\}_k \subseteq \mathcal{P}(X)$ such that

$$A_k := \left\{ x \in E \cap U : d(x, E \cap U^C) \geq \frac{1}{k} \right\} \subseteq E \cap U.$$

It is clear that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$, i.e. $\{A_k\}_k$ is an increasing sequence of sets. Also, $\bigcup_k A_k = E \cap U$.

[1] We look at points that get increasingly closer to the edge of the set $E \cap U$, or in other words, increasingly closer to $E \cap U^C$.

For each $k \in \mathbb{N}$, notice that $A_k \cup (E \cap U^C) \subseteq E$. Thus by subadditivity and additivity over disjoint sets, for every k, we have

$$\mu^*(E) \geq \mu^*(A_k \cup (E \cap U^C)) = \mu^*(A_k) + \mu^*(E \cap U^C).$$

Since $\{A_k\}_k$ is an increasing sequence of sets, it follows that

$$\mu^*(E) \geq \lim_{k \to \infty} \mu^*(A_k) + \mu^*(E \cap U^C).$$

Claim: $\lim_{k \to \infty} \mu^*(A_k) = \mu^*(E \cap U)$ Since $A_k \subseteq E \cap U$, by subadditivity,

$$\mu^*(A_k) \leq \mu^*(E \cap U).$$

It remains to prove the other inequality.

[2] Let $D_1 = A_1$, $D_2 = A_2 \setminus A_1$, $\ldots D_n = A_n \setminus A_{n-1}$. Then, notice that

$$\begin{aligned} E \cap U = \bigcup_n D_n &= A_1 \cup D_2 \cup D_3 \cup \ldots \\ &= A_2 \cup D_3 \cup D_4 \cup \ldots \\ &= A_n \cup D_{n+1} \cup D_{n+2} \cup \ldots, \end{aligned}$$

[2] This part here requires an escape from where we already are. If your head is in the muddle, stop reading, go out, walk, and then come back.

Here, we ask ourselves: so what if we look at how much the A_k's change as k increases?

since $\{A_n\}$ is an increasing sequence of sets. Now for $x \in D_n$, we have that $x \in A_n$ but $x \notin A_{n-1}$. [3] In particular, we have

$$\frac{1}{n} \leq d(x, E \cap U^C) < \frac{1}{n-1}.$$

[3] We see that the D_n's form some kind of a ring-like partitioning of $E \cup U$.

[4] Let $m \geq n+2$. Consider $y \in D_m$, $x \in D_n$ and $z \in E \cap U^C$. Then we know by the triangle inequality that

$$\frac{1}{n} \leq d(x, z) \leq d(x, y) + d(y, z).$$

[4] Let's look at putting every odd D_n's together, and see how far apart are they. Directly looking at D_n's altogether is difficult because then their boundaries get muddled together.

We may then pick $z_0 \in E \cap U^C$ such that

$$d(y, z_0) < \frac{1}{m-1}.$$

Then

$$\frac{1}{n} \leq d(x, z_0) < d(x, y) + \frac{1}{m-1},$$

and so

$$\frac{1}{n} - \frac{1}{m-1} < d(x, y).$$

Notice that

$$\frac{1}{n} - \frac{1}{m-1} \geq \frac{1}{n} - \frac{1}{n+2-1} = \frac{1}{n} - \frac{1}{n-1}.$$

Therefore, $\forall x \in D_n$ and $y \in D_m$, we have

$$\frac{1}{n} - \frac{1}{n-1} < d(x, y).$$

In other words,

$$d(D_n, D_m) > 0$$

as long as $m \geq n + 2$.

Since μ^* is a Metric Outer Measure, it follows that

$$\mu^*\left(\bigcup_{n \text{ odd}} D_n \right) = \sum_{n \text{ odd}} \mu^*(D_n).$$

Since $D_1 \cup D_3 \cup \ldots \subseteq E \cap U$, by subadditivity,

$$\mu^*(D_1 \cup D_3 \cup \ldots) \leq \mu^*(E \cap U) < \infty.$$

In particular,

$$\sum_{n \text{ odd}} \mu^*(D_n) < \infty.$$

Similarly, we can show that

$$\mu^*\left(\bigcup_{n \text{ even}} D_n \right) = \sum_{n \text{ even}} \mu^*(D_n) < \infty.$$

Putting the two together, we have

$$\mu^*\left(\bigcup_{n} D_n \right) = \sum_{n} \mu^*(D_n) < \infty.$$

Finally, since $E \cap U = A_n \cup D_{n+1} \cup D_{n+2} \cup \ldots$, by subadditivity,

$$\mu^*(E \cap U) \leq \mu^*(A_n) + \mu^*\left(\bigcup_{m>n} D_m\right) = \mu^*(A_n) + \sum_{m=n+1}^{\infty} \mu^*(D_m).$$

Since $\sum_n \mu^*(D_n) < \infty$, the tail

$$\sum_{m=n+1}^{\infty} \mu^*(D_m) \to 0$$

as $n \to \infty$. Therefore,

$$\begin{aligned} \mu^*(E \cap U) &\leq \lim_{n\to\infty} \mu^*(A_n) + \lim_{n\to\infty} \sum_{m=n+1}^{\infty} \mu^*(D_m) \\ &= \lim_{n\to\infty} \mu^*(A_n), \end{aligned}$$

as required. □

Proposition 10 (Lebesgue-Stieltjes Outer Measure on Half-open Intervals)

Let μ_F^* be the Lebesgue-Stieltjes Outer Measure. Then for $a < b \in \mathbb{R}$, we have

$$\mu_F^*((a, b]) = F(b) - F(a).$$

Proof

First, notice that $(a, b] \subseteq (a, b]$, and so

$$\mu_F^*((a, b]) \leq F(b) - F(a)$$

by definition.

Let $\varepsilon > 0$. Pick a covering $(a, b] \subseteq \bigcup_n (a_n, b_n]$ such that

$$\sum_n (F(b_n) - F(a_n)) \leq \mu_F^*((a, b]) + \varepsilon.$$

By right-continuity of F, we may pick $b_n' > b_n$ such that

$$F(b_n') < F(b_n) + \frac{\varepsilon}{2^n}.$$

Notice that

$$\begin{aligned}
\sum_n F(b_n') - F(a_n) &\leq \sum_n \left(F(b_n) + \frac{\varepsilon}{2^n} - F(a_n)\right) \\
&= \varepsilon + \sum_n (F(b_n) - F(a_n)) \\
&\leq \mu_F^*((a,b]) + 2\varepsilon.
\end{aligned}$$

Similarly, we can pick $a' > a$ such that $F(a') < F(a) + \varepsilon$. Then

$$[a, b] \subseteq (a, b] \subseteq \bigcup_n (a_n, b_n] \subseteq \bigcup_n (a_n, b_n').$$

By **compactness**, there exists a finite subcover, i.e. $\exists N \in \mathbb{N}$ such that

$$[a', b] \subseteq \bigcup_{k=1}^{N} (a_{n_k}, b_{n_k}'). \qquad \square$$

The proof shall be completed in the next lecture.

8 Lecture 8 Sep 20th 2019

8.1 The Lebesgue-Stieltjes Outer Measure (Continued 2)

Proof (Lebesgue-Stieltjes Outer Measure on Half-open Intervals continued)

Continuing from before, let us first reorder the finite number of intervals such that $b'_{n_1} \geq b'_{n_2} \geq \ldots$.

Figure 8.1 illustrates what sets do we throw away (labelled T), what we shall keep (labelled K), and what is impossible (labelled I).

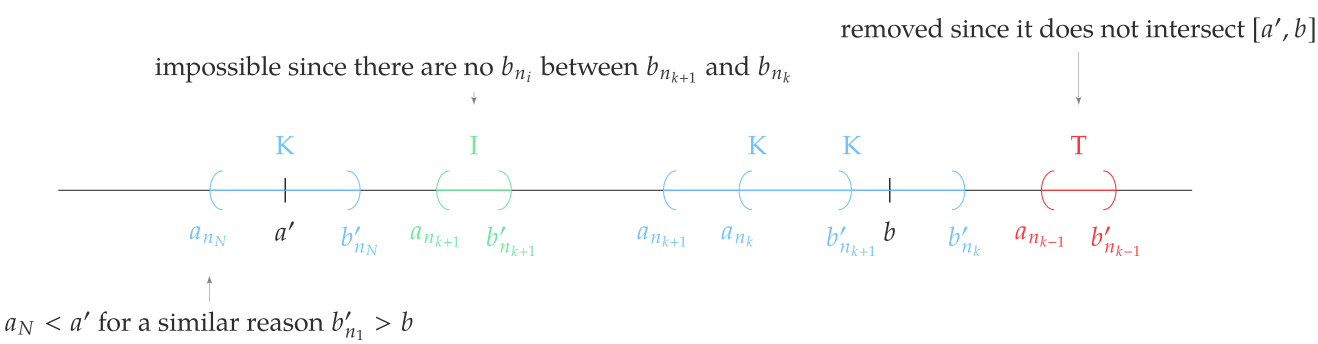

Figure 8.1: An arbitrary representation of the finite cover.

Most importantly, we observe that

$$a_{n_{k-1}} < b'_{n_k}.$$

Therefore,

$$\sum_{k=1}^{\infty} F(b'_{n_k}) - F(a_{n_k}) \geq \sum_{k=1}^{N} F(b'_{n_k}) - F(a_{n_k})$$

$$
\begin{aligned}
&= F(b_1') \overbrace{- F(a_1) + F(b_2')}^{>0} \overbrace{- F(a_2) + F(b_3')}^{>0} \\
&\quad \overbrace{+ \ldots + F(b_N')}^{>0} - F(a_N) \\
&\geq F(b_1') - F(a_N) \geq F(b) - F(a').
\end{aligned}
$$

It follows that

$$\mu_F^*((a,b]) + 2\varepsilon \geq F(b) - F(a') \geq F(b) - (F(a) + \varepsilon),$$

and so

$$F(b) - F(a) \leq \mu_F^*((a,b]) + 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, our proof is complete. □

Remark 8.1.1

Proposition 10 means that the Lebesgue-Stieltjes outer measure falls back nicely onto our usual notion of length when it comes to intervals.

Proposition 11 (The Lebesgue-Stieltjes Outer Measure is a Metric Outer Measure)

μ_F^* is a Metric Outer Measure.

Proof

Let $\delta > 0$. For each interval $(a,b] \subseteq \mathbb{R}$ such that $b - a > \delta$, we may break it up so that

$$(a,b] = (x_1, x_2] \cup (x_2, x_3] \cup \ldots \cup (x_{N-1}, x_N],$$

where $x_i - x_{i-1} < \delta$, and $x_1 = a$, $x_N = b$. Notice that

$$F(b) - F(a) = F(x_N) - F(x_{N-1}) + F(x_{N-1}) - \ldots + F_{x_2} - F(x_1).$$

Therefore, given $\delta > 0$, by the definition of μ_F^*, we have that $\forall E \subseteq \mathbb{R}$,

$$\mu_F^*(E) = \inf \left\{ \sum_{i=1}^{\infty} F(b_i) - F(a_i) : E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i],\ b_i - a_i < \delta \right\}.$$

Now let $A, B \subseteq \mathbb{R}$ such that $d(A, B) > 2\delta > 0$. Given any $\varepsilon > 0$, we can pick a cover $A \cup B \subseteq \bigcup_i (a_i, b_i]$ with $b_i - a_i < \delta$ such that

$$\sum_i F(b_i) - F(a_I) \leq \mu_F^*(A \cup B) + \varepsilon.$$

Since $d(A, B) > 2\delta$ and $b_i - a_i < \delta$ for each i, the following are the only possible scenarios: for each i,

- $A \cap (a_i, b_i] = \emptyset$ and $B \cap (a_i, b_i] = \emptyset$, in which case we choose an even finer covering of $A \cup B$ to remove $(a_i, b_i]$;
- $A \cap (a_i, b_i] \neq \emptyset$ and $B \cap (a_i, b_i] = \emptyset$; and
- $A \cap (a_i, b_i] = \emptyset$ and $B \cap (a_i, b_i] \neq \emptyset$.

We may thus consider the following subsets of indices:

$$\begin{aligned} \{i_k\}_{k \in \mathbb{N}} &= \{i_k : A \cap (a_{i_k}, b_{i_k}] \neq \emptyset,\ B \cap (a_{i_k}, b_{i_k}] = \emptyset\} \subseteq \{i\}_{i \in \mathbb{N}} \\ \{j_l\}_{l \in \mathbb{N}} &= \{j_l : A \cap (a_{j_l}, b_{j_l}] = \emptyset,\ B \cap (a_{j_l}, b_{j_l}] \neq \emptyset\} \subseteq \{i\}_{i \in \mathbb{N}}. \end{aligned}$$

In particular, we have

$$A \subseteq \bigcup_{k=1}^{\infty} (a_{i_k}, b_{i_k}] \text{ and } B \subseteq \bigcup_{l=1}^{\infty} (a_{j_l}, b_{j_l}].$$

Then by Proposition 10,

$$\begin{aligned} \mu_F^*(A) &\leq \sum_{k=1}^{\infty} F(b_{i_k}) - F(a_{i_k}) \\ \mu_F^*(B) &\leq \sum_{l=1}^{\infty} F(b_{j_l}) - F(a_{j_l}). \end{aligned}$$

It follows that

$$\mu_F^*(A) + \mu_F^*(B) \leq \sum_{k=1}^{\infty} F(b_{i_k}) - F(a_{i_k}) + \sum_{l=1}^{\infty} F(b_{j_l}) - F(a_{j_l})$$

$$= \sum_{i=1}^{\infty} F(b_i) - F(a_i) \leq \mu_F^*(A \cup B) + \varepsilon.$$

Since ε was arbitrary, we have

$$\mu_F^*(A) + \mu_F^*(B) \leq \mu_F^*(A \cup B)$$

as required. □

Theorem 12 (Lebesgue-Stieltjes Theorem by Carathéodory)

Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function that is right continuous. Let μ_F^* be the corresponding outer measure. Then the collection $\mathfrak{M}_F$ of μ_F^*-measurable sets contains $\mathfrak{B}(\mathbb{R})$ and $\mu_F : \mathfrak{M}_F \to [0, \infty]$ is a Complete Measure Space with

$$\mu_F((a, b]) = F(b) - F(a).$$

Proof

This is a direct result of Carathéodory's Second Theorem, Proposition 10, and Proposition 11. □

Example 8.1.1

When $F(x) = x$, μ_F is simply Lebesgue's measure.

Example 8.1.2 (Dirac delta measure of a point)

Fix $x_0 \in \mathbb{R}$. Let

$$F(x) = \begin{cases} 0 & x < x_0 \\ 1 & x \geq x_0 \end{cases}.$$

Notice that

$$\forall b > x_0 \quad \mu_F((x_0, b]) = F(b) - F(x_0) = 1 - 1 = 0,$$

which then

$$\mu_F((x_0, \infty]) = 0.$$

Also

$$\forall a < x_0 \quad \mu_F\left(\left(a, x_0 - \frac{1}{n}\right]\right) = F\left(x_0 - \frac{1}{n}\right) - F(a) = 0 - 0 = 0,$$

which then since

$$(a, x_0) = \bigcup_{n=1}^{\infty} \left(a, x_0 - \frac{1}{n}\right],$$

we have

$$\mu_F((a, x_0)) = 0,$$

which since this holds for all $a < x_0$,

$$\mu_F((-\infty, x_0)) = 0.$$

However, for $a < x_0$,

$$\mu_F((a, x_0]) = F(x_0) - F(a) = 1 - 0 = 1.$$

Furthermore, since

$$\{x_0\} = \bigcap_{n=1}^{\infty} \left(x_0 - \frac{1}{n}, x_0\right],$$

by continuity from above,

$$\mu_F(\{x_0\}) = \lim_{n \to \infty} \mu_F\left(x_0 - \frac{1}{n}, x_0\right] = 1.$$

With the above example in mind, recall the Cantor set

$$C = \bigcap_{n=1}^{\infty} C_n,$$

where $C_n = C_{n-1} \setminus P_n$, where P is the middle $1/3$ of each of the remaining intervals, with $C_0 = [0, 1]$.

We have that the Lebesgue measure of each C_n is

$$\mu(C_1) = \frac{1}{3}$$
$$\mu(C_2) = \mu(C_1) - \frac{2}{9} = 1 - \frac{1}{3} - \frac{2}{9}$$

$$\mu(C_3) = \frac{1}{3} - \frac{2}{9} - \frac{4}{27}$$
$$\vdots$$
$$\mu(C_n) = 1 - \frac{1}{3} - \frac{2}{9} - \ldots - \frac{2^{n-1}}{3^n}.$$

Then

$$\mu(C) = 1 - \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1 - \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} = 1 - \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 0.$$

Cantor Function With the Cantor set, we may construct the famous/infamous **Cantor function**. The Cantor function, which we shall label F, is defined such that F is

- $\frac{1}{2}$ on $\left(\frac{1}{3}, \frac{2}{3}\right)$,
- $\frac{1}{4}$ on $\left(\frac{1}{9}, \frac{2}{9}\right)$,
- $\frac{3}{4}$ on $\left(\frac{7}{9}, \frac{8}{9}\right)$,

and so on, on each of the removed intervals. We also let $F(0) = 0$ and $F(1) = 1$. Then F is increasing and continuous. Furthermore, $F' = 0$ on all the removed intervals.

9 Lecture 9 Sep 23rd 2019

9.1 The Lebesgue-Stieltjes Outer Measure (Continued 3)

Continuing with the Cantor set, we notice that $F'(x) = 0$ for all $x \notin C$. In particular, the derivative of F exists almost everywhere.

Now for intervals that we have "thrown away", by Example 8.1.2, the measure of each of these intervals is 0. However,

$$\mu_F[0,1] = F(1) - F(0) = 1.$$

Since $[0,1] = C \cup C^C$, we have that

$$1 = \mu_F[0,1] = \mu_F(C) + \mu_F(C^C) = \mu_F(C) + 0,$$

and so

$$\mu_F(C) = 1.$$

Remark 9.1.1

On $\mathbb{R}^k$, we can define a k-dimensional Lebesgue outer measure μ_F^* by covering sets with "boxes" such as

$$R_k = (a_1, b_1) \times \ldots \times (a_k, b_k),$$

where

$$\mathrm{Vol}(R_k) = (b_1 - a_1) \ldots (b_k - a_k).$$

Thus

$$\mu_k^*(E) := \inf \left\{ \sum \mathrm{Vol}(R_k) : E \subseteq \bigcup_k R_k \right\}.$$

Exercise 9.1.1

Show that $\mu_k^*(E)$ *for each* $k \geq 2$ *is a metric outer measure.*

Applying Carathéodory's theorems, we get the k-dimensional measure, and we know that $\mathfrak{B}(\mathbb{R}^k)$ *are all measurable.*

Theorem 13 (A Measure Constructed By Another Measure)

Let $\mu : \mathfrak{B}(\mathbb{R}) \to [0, \infty]$ *be a measure with*

$$\forall a, b \in \mathbb{R} \quad \mu((a, b]) < \infty.$$

Define

$$F(x) := \begin{cases} \mu((0, x]) & x > 0 \\ 0 & x = 0 \\ -\mu((x, 0]) & x < 0 \end{cases}.$$

Then F *is increasing and right-continuous. Furthermore,* $\mu(A) = \mu_F(A)$ *for all* $A \in \mathfrak{B}(\mathbb{R})$.

Proof

F is increasing

$0 \leq x < y$

Notice that $(0, x] \subseteq (0, y]$. Thus by subadditivity,

$$F(x) = \mu(0, x] \leq \mu(0, y] = F(y).$$

$x < y \leq 0$

Observe that

$$(x, 0] \supseteq (y, 0],$$

and so subadditivity dictates that

$$\mu(x, 0] \geq \mu(y, 0].$$

Thus

$$F(x) = -\mu(x, 0] \leq -\mu(y, 0] = F(y).$$

F is right-continuous

$0 \leq x$

Consider a sequence $x_n \searrow x$. [1] Then notice that

$$(0, x] = \bigcap_{n=1}^{\infty} (0, x_n],$$

and

$$(0, x_n] \supseteq (0, x_{n+1}],$$

i.e. $\{(0, x_n]\}_n$ is a decreasing sequence of sets. Furthermore, we note that

$$\mu(0, x_n] < \infty$$

by assumption. Thus, by continuity from above,

$$F(x) = \mu(0, x] = \lim_{n \to \infty} \mu(0, x_n] = \lim_{n \to \infty} F(x_n).$$

[1] We write $x_n \searrow x$ to mean that $\{x_n\}_n$ is a sequence such that $x < x_n$ and $\lim_{n\to\infty} x_n = x$.

$x < 0$

Consider a sequence $x_n \searrow x$. Then, notice that

$$(x, 0] = \bigcup_n (x_n, 0],$$

and

$$(x_n, 0] \subseteq (x_{n+1}, 0],$$

I.e. $\{(x_n, 0]\}_n$ is an increasing sequence of sets. By continuity from below,

$$F(x) = -\mu(x, 0] = \lim_{n \to \infty} -\mu(x_n, 0] = \lim_{n \to \infty} F(x_n).$$

$\forall A \in \mathfrak{B}(\mathbb{R}) \quad \mu(A) = \mu_F(A)$ Consider the set

$$\mathcal{A} := \{A \in \mathfrak{B}(\mathbb{R}) : \mu(A) = \mu_F(A)\} \subseteq \mathfrak{B}(\mathbb{R}).$$

Now if $\mathcal{A}$ is a σ-algebra and contains intervals, then $\mathfrak{B}(\mathbb{R}) \subseteq \mathcal{A}$, which means $\mathfrak{B}(\mathbb{R}) = \mathcal{A}$, which is equivalent to what we want.

$\mathcal{A}$ contains intervals

- Let $0 \leq a < b \in \mathbb{R}$. Notice that

$$\mu(0, a] + \mu(a, b] = \mu(0, b],$$

since

$$(0,a] \,\dot\cup\, (a,b] = (0,b].$$

Since $F(a) = \mu(0,a]$ and $F(b) = \mu(0,b]$, it follows that

$$F(a) + \mu(a,b] = F(b),$$

and so by Proposition 10,

$$\mu(a,b] = F(b) - F(a) = \mu_F(a,b].$$

- Let $a < 0 \leq b \in \mathbb{R}$. Notice that

  $$(a,b] = (a,0] \,\dot\cup\, (0,b],$$

  and so by Proposition 10,

  $$\mu(a,b] = \mu(a,0] + \mu(0,b] = -F(a) + F(b) = \mu_F(a,b].$$

- Let $a < b \leq 0 \in \mathbb{R}$. Notice that

  $$(a,0] = (a,b] \,\dot\cup\, (b,0],$$

  and so

  $$\mu(a,0] = \mu(a,b] + \mu(b,0].$$

  By Proposition 10,

  $$\mu(a,b] = \mu(a,0] - \mu(b,0] = -F(a) + F(b) = \mu_F(a,b].$$

$\mathcal{A}$ is a σ-algebra

- It is clear that

  $$\mu(\emptyset) = 0 = \mu_F(\emptyset),$$

  and so $\emptyset \in \mathcal{A}$.

- Let $\{A_n\}_n \subseteq \mathcal{A}$ be a disjoint collection. Then it is clear that

  $$\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n) = \sum_n \mu_F(A_n) = \mu_F\left(\bigcup_n A_n\right).$$

- Let $A \in \mathcal{A}$. Consider an interval $(a, b] \subseteq \mathcal{A}$. Note that

  $$(a, b] = ((a, b] \cap A) \cup ((a, b] \cap A^C).$$

  So

  $$\mu(a, b] = \mu((a, b] \cap A) + \mu((a, b] \cap A^C). \tag{9.1}$$

  Consider an arbitrary covering

  $$(a, b] \cap A \subseteq \bigcup_i (a_i, b_i].$$

  Then

  $$\mu((a, b] \cap A) \leq \sum_i \mu(a_i, b_i] = \sum_i F(b_i) - F(a_i).$$

  Thus

  $$\begin{aligned} \mu((a, b] \cap A) &\leq \inf \left\{ \sum_i F(b_i) - F(a_i) : (a, b] \cap A \subseteq \bigcup_i (a_i, b_i) \right\} \\ &= \mu_F^*((a, b] \cap A) = \mu_F((a, b] \cap A). \end{aligned}$$

  Similarly, we have

  $$\mu((a, b) \cap A^C) \leq \mu_F((a, b] \cap A^C).$$

  Therefore, going back to Equation (9.1),

  $$\begin{aligned} \mu(a, b] &\leq \mu_F((a, b] \cap A) + \mu_F((a, b] \cap A^C) \\ &= \mu_F(a, b] = \mu(a, b]. \end{aligned}$$

  It follows that we must have

  $$\mu((a, b] \cap A) = \mu_F((a, b] \cap A),$$

  and

  $$\mu((a, b] \cap A^C) = \mu_F((a, b] \cap A^C).$$

  In particular, $(a, b] \cap A^C \in \mathcal{A}$. Notice that

  $$A^C = \dot{\bigcup_{n=-\infty}^{\infty}} (n, n+1] \cap A^C,$$

  and so since we've showed that $\mathcal{A}$ is closed under countable unions, $A^C \in \mathcal{A}$.

This concludes the proof. □

9.2 Measurable Functions

We look into so-called measurable functions that shall be our next step towards the theory of integration.

Definition 17 (Measurable Space)

Let $X \neq \emptyset$ be a set and $\mathfrak{M}$ a σ-algebra of subsets of X. We call the pair $(X, \mathfrak{M})$ a ***measurable space****.*

Definition 18 (Measurable Functions)

Let $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ be measurable spaces, and $f : X \to Y$ a function. We say that f is a ***$(\mathfrak{M}, \mathfrak{N})$-measurable function****, or that f is* ***$(\mathfrak{M}, \mathfrak{N})$-measurable****, if*

$$\forall E \in \mathfrak{N} \quad f^{-1}(E) \in \mathfrak{M}.$$

Remark 9.2.1

For those who remember contents from real analysis, the definition of a measurable function is similar to the definition of a continuous function on topological spaces. We shall, in fact, see that their similarity goes beyond than their definitions.

Proposition 14 (Composition of Measurable Functions)

Let $(X, \mathfrak{M})$, $(Y, \mathfrak{N})$ and $(Z, \mathfrak{O})$ be measurable spaces. Suppose

- *$f : X \to Y$ is $(\mathfrak{M}, \mathfrak{N})$-measurable; and*
- *$g : Y \to Z$ is $(\mathfrak{N}, \mathfrak{O})$-measurable.*

Then $g \circ f : X \to Z$ is $(\mathfrak{M}, \mathfrak{O})$-measurable.

Proof

Let $E \in \mathfrak{O}$. Then since $g^{-1}(E) \in \mathfrak{N}$, it follows that

$$(g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E)) \in \mathfrak{M}. \qquad \square$$

Proposition 15 (Measurability of a Function Defined on Generators of the Codomain)

Let $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ be measurable spaces, and let $\mathcal{E}$ be the generator of $\mathfrak{N}$. Let $f : X \to Y$. If $f^{-1}(E) \in \mathfrak{M}$ for all $E \in \mathcal{E}$, then f is $(\mathfrak{M}, \mathfrak{N})$-measurable.

Proof

Consider

$$\mathcal{A} := \{A \in \mathfrak{N} : f^{-1}(A) \in \mathfrak{M}\}.$$

Notice that if $\mathcal{A}$ is a σ-algebra, then we must have $\mathcal{E} \subseteq \mathcal{A}$, which then forces $\mathcal{A} = \mathfrak{N}$.

$\mathcal{A}$ is a σ-algebra

- Since $\mathcal{E}$ is a σ-algebra, $\emptyset \in \mathcal{E}$, and so $f^{-1}(\emptyset) \in \mathfrak{M}$ by assumption. Thus $\emptyset \in \mathcal{A}$.

- Suppose $\{A_n\}_n \subseteq \mathcal{A}$ a disjoint collection. Notice that since f is a function, it must be that [2]

$$f^{-1}\left(\bigcupdot_n A_n \right) = \bigcupdot_n f^{-1}(A_n).$$

[2] If this is not clear, notice that if some $f^{-1}(A_n)$ and $f^{-1}(A_m)$ are not disjoint, then that means A_n and A_m are not disjoint, or that f is not a function.

Since $f^{-1}(A_n) \in \mathfrak{M}$ and $\mathfrak{M}$ is a σ-algebra, it follows that

$$f^{-1}\left(\dot{\bigcup_n} A_n\right) = \dot{\bigcup_n} f^{-1}(A_n) \in \mathfrak{M}.$$

Thus $\bigcup_n A_n \in \mathcal{A}$.

- Suppose $A \in \mathcal{A}$. Notice that $f^{-1}(A) \in \mathfrak{M}$, and so $(f^{-1}(A))^C \in \mathfrak{M}$. We need to show that

$$f^{-1}(A^C) = (f^{-1}(A))^C$$

But this follows for the same reason as the last point.

It follows that $\mathcal{A}$ is a σ-algebra. □

Corollary 16 (Continuous Functions on Borel Sets are Measurable)

Let X and Y be topological spaces, with $\mathfrak{B}(X)$ and $\mathfrak{B}(Y)$ as their corresponding Borel sets. Suppose $f : X \to Y$ is continuous. Then f is $(\mathfrak{B}(X), \mathfrak{B}(Y))$-measurable.

Proof

Let $U \in \mathfrak{B}(Y)$, i.e. $U \subseteq Y$ is an open set. Since f is continuous, $f^{-1}(U)$ is open in X, i.e. $f^{-1}(U) \in \mathfrak{B}(X)$. Thus f is $(\mathfrak{B}(X), \mathfrak{B}(Y))$-measurable. Furthermore,

$$\mathcal{E} := \{U \subseteq Y : U \text{ is open }\}$$

generates $\mathfrak{B}(Y)$. □

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10.1 Measurable Functions (Continued)

Notation

Let $(X, \mathfrak{M})$ be a measurable space. The function $f : X \to \mathbb{R}$ is said to be $\mathfrak{M}$-measurable, or measurable, when it is $(\mathfrak{M}, \mathfrak{B}(\mathbb{R}))$-measurable.

Proposition 17 (Characteristics of $\mathfrak{M}$-measurable Functions)

Let $(X, \mathfrak{M})$ be a measurable space and $f : X \to \mathbb{R}$. TFAE:

1. *f is $\mathfrak{M}$-measurable.*
2. $\forall a \in \mathbb{R} \quad f^{-1}((a, \infty)) \in \mathfrak{M}$.
3. $\forall a \in \mathbb{R} \quad f^{-1}([a, \infty)) \in \mathfrak{M}$.
4. $\forall a \in \mathbb{R} \quad f^{-1}((-\infty, a)) \in \mathfrak{M}$.
5. $\forall a \in \mathbb{R} \quad f^{-1}((-\infty, a]) \in \mathfrak{M}$.

Proof

We shall only look at $(1) \iff (2)$, since the proof for $(1) \iff (i)$ for $i = 3, 4, 5$ are similar.

$(1) \implies (2)$ $\forall a \in \mathbb{R}$, since $(a, \infty) \in \mathfrak{B}(\mathbb{R})$, it follows by $\mathfrak{M}$-measurability

of f that
$$f^{-1}(a,\infty) \in \mathfrak{M}.$$

(2) $\implies$ (1) We know that

$$\mathcal{E} := \{(a,\infty) : a \in \mathbb{R}\}$$

generates $\mathfrak{B}(\mathbb{R})$. By assumption $\forall E \in \mathcal{E}$, $f^{-1}(E) \in \mathfrak{M}$. By Proposition 15, it follows that f is indeed a $\mathfrak{M}$-measurability. □

Remark 10.1.1

When $X = \mathbb{R}$, we say that $f : \mathbb{R} \to \mathbb{R}$ is measurable

$$\iff \forall B \in \mathfrak{B}(\mathbb{R}) \quad f^{-1}(B) \in \mathfrak{B}(\mathbb{R})$$
$$\iff \forall a \in \mathbb{R} \quad f^{-1}(a,\infty) \in \mathfrak{B}(\mathbb{R}).$$

Let $\mathcal{L}$ be the σ-algebra of all Lebesgue measurable sets. $f : \mathbb{R} \to \mathbb{R}$ is said to be Lebesgue measurable when

$$\forall a \in \mathbb{R} \quad f^{-1}(a,\infty) \in \mathcal{L}$$
$$\iff \forall B \in \mathfrak{B}(\mathbb{R}) \quad f^{-1}(B) \in \mathcal{L}.$$

Warning

Notice that the last remark can be problematic. Compare what was written above with Proposition 14. In particular, notice that for the definition of a Lebesgue measurable function, instead of requiring $f^{-1}(a,\infty) \in \mathfrak{B}(\mathbb{R})$, we simply required $f^{-1}(a,\infty) \in \mathcal{L}$. Thus, if we have another function $g : \mathbb{R} \to \mathbb{R}$, for $f \circ g$ to be Lebesgue measurable, we require

$$(f \circ g)^{-1}(a,\infty) \in \mathcal{L}.$$

However, $f^{-1}(a,\infty) \in \mathcal{L}$, and it is not necessarily true that

$$g^{-1}(f^{-1}(a,\infty)) \in \mathcal{L}.$$

There are various examples that show this, typically arising from the Cantor Function.

We go on a little tangent about products of σ-algebras to make our lives down the road a little easier.

Definition 19 (Products of σ-algebras)

Let $(Y_1, \mathfrak{N}_1)$ and $(Y_2, \mathfrak{N}_2)$ be measurable spaces. We define $\mathfrak{N}_1 \otimes \mathfrak{N}_2$ to be the σ-algebra on the Cartesian product $Y_1 \times Y_2$ as

$$\mathfrak{N}_1 \otimes \mathfrak{N}_2 := \{B_1 \times B_2 : B_1 \in \mathfrak{N}_1,\ B_2 \in \mathfrak{N}_2\}.$$

Remark 10.1.2

We will unofficially call $\mathfrak{N}_1 \otimes \mathfrak{N}_2$ the ***tensor product of $\mathfrak{N}_1$ and $\mathfrak{N}_2$****.*

Proposition 18 (Tensor Product of $\mathfrak{B}(\mathbb{R})$'s)

We have

$$\mathfrak{B}(\mathbb{R}) \otimes \mathfrak{B}(\mathbb{R}) = \mathfrak{B}(\mathbb{R}^2).$$

Proof

$\mathfrak{B}(\mathbb{R}^2) \subseteq \mathfrak{B}(\mathbb{R}) \otimes \mathfrak{B}(\mathbb{R})$ Let $\mathcal{O} \subseteq \mathbb{R}^2$ be open. Then

$$\mathcal{O} = \cup\{(r_1, s_1) \times (r_2, s_2) \subseteq \mathcal{O} : r_1, r_2, s_1, s_2 \in \mathbb{Q}\},$$

which means $\mathcal{O}$ is a countable union of open sets. Since $(r_1, s_1), (r_2, s_2) \in \mathfrak{B}(\mathbb{R})$, it follows that

$$(r_1, s_1) \times (r_2, s_2) \in \mathfrak{B}(\mathbb{R}) \otimes \mathfrak{B}(\mathbb{R}),$$

and so $\mathcal{O} \in \mathfrak{B}(\mathbb{R}) \otimes \mathfrak{B}(\mathbb{R})$ since $\mathfrak{B}(\mathbb{R}) \otimes \mathfrak{B}(\mathbb{R})$ is a σ-algebra.

$\mathfrak{B}(\mathbb{R}) \otimes \mathfrak{B}(\mathbb{R}) \subseteq \mathfrak{B}(\mathbb{R}^2)$ WTS

$$\forall B_1, B_2 \in \mathfrak{B}(\mathbb{R}) \quad B_1 \times B_2 \in \mathfrak{B}(\mathbb{R}^2).$$

Let

$$\mathcal{A} := \{E \subseteq \mathbb{R} : E \times \mathbb{R} \in \mathfrak{B}(\mathbb{R}^2)\}.$$

If $\mathcal{A}$ is a σ-algebra and $(a, b) \in \mathcal{A}$ for any $a < b \in \mathbb{R}$, then $\mathfrak{B}(\mathbb{R}) \subseteq \mathcal{A}$ and we are done.

$\mathcal{A}$ is a σ-algebra

- $\emptyset \times \mathbb{R} = \mathbb{R} \in \mathfrak{B}(\mathbb{R})$ and so $\emptyset \in \mathcal{A}$.
- Suppose $\{E_n\}_n \subseteq \mathcal{A}$ is a disjoint collection of sets. Then $E_n \times \mathbb{R} \in \mathfrak{B}(\mathbb{R}^2)$ for each n. Since $\mathfrak{B}(\mathbb{R}^2)$ is a σ-algebra, it follows that

  $$\dot{\bigcup_n}(E_n \times \mathbb{R}) \in \mathfrak{B}(\mathbb{R}^2).$$

  Notice that

  $$\dot{\bigcup_n}(E_n \times \mathbb{R}) = \left(\dot{\bigcup_n} E_n\right) \times \mathbb{R}.$$

  Therefore $\dot{\bigcup}_n E_n \in \mathcal{A}$.
- Let $E \in \mathcal{A}$. Then

  $$E^C \times \mathbb{R} = (E \times \mathbb{R})^C \in \mathfrak{B}(\mathbb{R}^2).$$

  Thus $E^C \in \mathcal{A}$.

$\forall a, b \in \mathbb{R} \quad (a, b) \times \mathbb{R} \in \mathcal{A}$ This is indeed true since $(a, b) \times \mathbb{R}$ is open.

Similarly, we can do the same for

$$\tilde{\mathcal{A}} := \{F \subseteq \mathbb{R} : \mathbb{R} \times F \in \mathfrak{B}(\mathbb{R}^2)\},$$

and have $\mathfrak{B}(\mathbb{R}) \subseteq \tilde{\mathcal{A}}$.

Let $B_1, B_2 \in \mathfrak{B}(\mathbb{R})$. So $B_1 \in \mathcal{A}$ and $B_2 \in \tilde{\mathcal{A}}$, and

$$B_1 \times \mathbb{R} \in \mathfrak{B}(\mathbb{R}^2) \text{ and } \mathbb{R} \times B_2 \in \mathfrak{B}(\mathbb{R}^2).$$

Therefore

$$B_1 \times B_2 = (B_1 \times \mathbb{R}) \cap (\mathbb{R} \times B_2) \in \mathfrak{B}(\mathbb{R}^2). \qquad \square$$

Proposition 19 (Component-wise Measurability of Functions)

Let $(X, \mathfrak{M})$, $(Y_1, \mathfrak{N}_1)$ and $(Y_2, \mathfrak{N}_2)$ be measurable functions. Let $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$. Let $f : X \to (Y_1, Y_2)$ such that

$$f(x) = (f_1(x), f_2(x)).$$

Then f is $(\mathfrak{M}, (\mathfrak{N}_1, \mathfrak{N}_2))$-measurable $\iff$ f_1 is $(\mathfrak{M}, \mathfrak{N}_1)$-measurable and f_2 is $(\mathfrak{M}, \mathfrak{N}_2)$-measurable.

Proof

$(\implies)$ Let $B_1 \in \mathfrak{N}_1$. WTS $f_1^{-1}(B_1) \in \mathfrak{M}$. We know $B_1 \times Y_2 \in \mathfrak{N}_1 \otimes \mathfrak{N}_2$, and $f^{-1}(B_1 \times Y_2) \in \mathfrak{M}$ since f is $(\mathfrak{M}, (\mathfrak{N}_1, \mathfrak{N}_2))$-measurable. Then

$$\begin{aligned} x \in f^{-1}(B_1 \times Y_2) &\iff f(x) \in B_1 \times Y_2 \\ &\iff (f_1(x), f_2(x)) \in B_1 \times Y_2 \\ &\iff f_1(x) \in B_1 \\ &\iff x \in f_1^{-1}(B_1). \end{aligned}$$

Thus $f_1^{-1}(B_1) = f^{-1}(B_1 \times Y_2) \in \mathfrak{M}$. Hence f_1 is $(\mathfrak{M}, \mathfrak{N}_1)$-measurable.

The proof is similar for f_2 being $(\mathfrak{M}, \mathfrak{N}_2)$-measurable.

$(\impliedby)$ [1] Let

$$\mathcal{A} := \{B \subseteq Y_1 \times Y_2 : f^{-1}(B) \in \mathfrak{M}\}.$$

[1] Again, we use the trick of showing that a cleverly chosen set that has the property that we want is a σ-algebra.

Notice that f is $(\mathfrak{M}, (\mathfrak{N}_1, \mathfrak{N}_2))$-measurable iff $\mathfrak{N}_1 \otimes \mathfrak{N}_2 \subseteq \mathcal{A}$.

$\mathcal{A}$ is a σ-algebra

- Let $B \in \mathcal{A}$. Then $f^{-1}(B)^C \in \mathfrak{M}$. Thus

$$\begin{aligned} x \in f^{-1}(B)^C &\iff x \notin f^{-1}(B) \\ &\iff f(x) \notin B \\ &\iff f(x) \in B^C \\ &\iff x \in f^{-1}(B^C). \end{aligned}$$

  Thus $f^{-1}(B^C) = f^{-1}(B)^C \in \mathfrak{M}$.

- Suppose $\{B_n\}_n \subseteq \mathcal{A}$. Consider $\{C_n\}_n \subseteq \mathcal{A}$ where $C_n = B_n \setminus \bigcup_{j=1}^{n-1} B_j$ and $C_1 = B_1$. Notice that $C_n \in \mathfrak{M}$ for each n. Also, $\{C_n\}_n$

is pairwise disjoint. It follows that

$$f^{-1}\left(\bigcup_n B_n\right) = f^{-1}\left(\dot{\bigcup_n} C_n\right) = \dot{\bigcup_n} f^{-1}(C_n) \in \mathfrak{M}.$$

This completes the claim.

$\mathfrak{N}_1 \otimes \mathfrak{N}_1 \subseteq \mathcal{A}$ WTS $\forall B_1 \in \mathfrak{N}_1 \;\; \forall B_2 \in \mathfrak{N}_2 \;\; B_1 \times B_2 \in \mathcal{A}$. We know that this is true iff

$$\forall B_1 \in \mathfrak{N}_1 \;\; \forall B_2 \in \mathfrak{N}_2 \;\; f^{-1}(B_1 \times B_2) \in \mathfrak{M}.$$

Notice that

$$\begin{aligned} x \in f_1^{-1}(B_1) \wedge x \in f_2^{-1}(B_2) &\iff f_1(x) \in B_1 \wedge f_2(x) \in B_2 \\ &\iff (f_1(x), f_2(x)) \in B_1 \times B_2 \\ &\iff f(x) \in B_1 \times B_2 \\ &\iff x \in f^{-1}(B_1 \times B_2). \end{aligned}$$

Thus $f^{-1}(B_1 \times B_2) = f_1^{-1}(B_1) \cap f_2^{-1}(B_2) \in \mathfrak{M}$. □

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Homework (Homework 7)

Let F be the cumulative distribution function (cdf) for the flip of a fair coin. Prove that $\mathfrak{M}_F = \mathcal{P}(\mathbb{R})$*. Find and prove a formula for* $\mu_F(A)$ *for any* $A \subseteq \mathbb{R}$.

Homework (Homework 8)

Let

$$F_{x_0}(t) = \begin{cases} 0 & t < x_0 \\ 1 & x_0 \leq t \end{cases}.$$

Let $\{r_n\}_n$ *be an enumeration of* $\mathbb{Q}$*. Set*

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} F_{r_n}(x).$$

Prove

1. *F is strictly increasing (I.e.* $x < y \implies F(x) < F(y)$*) and right continuous.*
2. *Find a prove a formula for* $\mu_F^*(A)$ *for any* $A \in \mathfrak{M}_F$.
3. *Prove* $\mathfrak{M}_F = \mathcal{P}(\mathbb{R})$.

Homework (Homework 9)

Let $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ be measurable spaces. For $E \subseteq X \times Y$, set

$$E_x := \{y \in Y : (x, y) \in E\}.$$

If $E \in \mathfrak{M} \otimes \mathfrak{N}$, prove $E_x \in \mathfrak{N}$.

11.1 Measurable Functions (Continued 2)

Definition 20 (Extended Reals)

We define the ***extended real numbers*** *as*

$$\mathbb{R}_e := \mathbb{R} \cup \{-\infty, \infty\}.$$

Note 11.1.1

The Borel set of $\mathbb{R}_e$ is

$$\mathfrak{B}(\mathbb{R}_e) = \{B \subseteq \mathbb{R}_e : B \cap \mathbb{R} \in \mathfrak{B}(\mathbb{R})\}.$$

Proposition 20 (Characteristics of $(\mathfrak{M}, \mathfrak{B}(\mathbb{R}_e))$-measurable Functions)

Let $(X, \mathfrak{M})$ be measurable. Let $f : X \to \mathbb{R}_e$. TFAE:

1. *f is $(\mathfrak{M}, \mathfrak{B}(\mathbb{R}_e))$-measurable.*
2. $\forall a \in \mathbb{R} \;\; f^{-1}(a, \infty] \in \mathfrak{M}$.
3. $\forall a \in \mathbb{R} \;\; f^{-1}[a, \infty] \in \mathfrak{M}$.
4. $\forall a \in \mathbb{R} \;\; f^{-1}[-\infty, a) \in \mathfrak{M}$.
5. $\forall a \in \mathbb{R} \;\; f^{-1}[-\infty, a] \in \mathfrak{M}$.
6. $\forall a \in \mathbb{R} \;\; f^{-1}(a, \infty) \in \mathfrak{M}$ *and* $f^{-1}(\{\infty\}), f^{-1}(\{-\infty\}) \in \mathfrak{M}$.

When any of the above hold, we say that f is measurable to mean that f is $(\mathfrak{M}, \mathfrak{B}(\mathbb{R}_e))$-measurable.

Proof

The proof is similar to that of Proposition 17. $\square$

Proposition 21 (Extremas, Supremas and Infimas of Measurable Functions)

Let $(X, \mathfrak{M})$ be measurable, and $\{f_j : X \to \mathbb{R}_e\}_j$ be a sequence of countable or finite number of functions. If each f_j is measurable, then

1. $g_1(x) = \sup_j f_j(x)$,
2. $g_2(x) = \inf_j f_j(x)$,
3. $g_3(x) = \limsup_j f_j(x)$, *and*
4. $g_4(x) = \liminf_j f_j(x)$

are all measurable.

Proof

We shall prove for (1) and (3), since the proof of (2) and (4) follow similarly, respectively.

1. Let $a \in \mathbb{R}$. Notice that

$$\begin{aligned} g_1(x) > a &\iff \exists j_0 \;\; f_{j_0}(x) > a \\ &\iff x \in \bigcup_{j=1}^{\infty} f_j^{-1}(a, \infty]. \end{aligned}$$

It follows that

$$g_1^{-1}(a, \infty] = \bigcup_{j=1}^{\infty} f_j^{-1}(a, \infty] \in \mathfrak{M}.$$

Hence g_1 is measurable.

3. Let $h_k(x) = \sup_{j \geq k} f_j(x)$. Then $\{h_k\}_k$ is a decreasing sequence. Note that

$$g_3(x) = \limsup_j f_j(x) = \lim_k h_k(x) = \inf_k h_k(x).$$

By (1), we know that each h_k is measurable. By (2), we know that $\inf_k h_k$ is measurable. Hence, g_3 is measurable as desired. □

Corollary 22 (Min and Max Functions are Measurablee)

Let $(X, \mathfrak{M})$ *be a measurable space, and* $f_1, f_2 : X \to \mathbb{R}_e$. *Then*

$$\max\{f_1, f_2\} \textit{ and } \min\{f_1, f_2\}$$

are both measurable.

Proof

Note that

$$\max\{f_1, f_2\} = \sup\{f_1, f_2\}$$

and

$$\min\{f_1, f_2\} = \inf\{f_1, f_2\}.$$

The result follows by Proposition 21. □

Corollary 23 (Limit points of a Sequence of Measurable Functions forms a Measurable Set)

Let $(X, \mathfrak{M})$ *be a measurable space. Let* $\{f_j : X \to \mathbb{R}\}_j$. *Let*

$$E = \{x \in X : \lim_j f_j(x) \textit{ exists } \}.$$

Then $E \in \mathfrak{M}$.

Proof

First, note that $\lim_j f_j(x)$ exists iff

$$g_3(x) = \limsup_j f_j(x) = \liminf_j f_j(x) = g_4(x).$$

Thus,

$$\begin{aligned} E &:= \{x \in X : \lim_j f_j(x) \text{ exists } \} \\ &= \{x \in X : g_3(x) = g_4(x)\} \\ &= \{x \in X : (g_3 - g_4)(x) = 0\} \\ &= (g_3 - g_4)^{-1}(\{0\}) \in \mathfrak{M}. \end{aligned}$$

□

Definition 21 (f^+ and f^-)

Let $f : X \to \mathbb{R}_e$. We define

$$f^+(x) := \max\{f(x), 0\},$$

and

$$f^-(x) := \max\{-f(x), 0\}.$$

Remark 11.1.1

1. *We see that $f = f^+ - f^-$.*
2. [1] *If f is measurable, then f^+ and f^- are measurable.*
3. $|f| = f^+ + f^-$.
4. $f^+ \cdot f^- = 0$.

Recall the Characteristic Function.

[1] In what is possibly a statement similar to Proposition 19, we can show this.

Proposition 24 (Characteristic Function of Measurable Sets are Measurable)

Let $(X, \mathfrak{M})$ be a measurable set. Suppose $A \subseteq X$. Then

$$\chi_A \text{ is measurable } \iff A \in \mathfrak{M}.$$

Proof

Notice that

- $1 < a \implies \chi_A^{-1}(a, \infty) = \emptyset \in \mathfrak{M}$.
- $0 \leq a \leq 1 \implies \chi_A^{-1}(a, \infty) = A$.
- $a < 0 \implies \chi_A^{-1}(a, \infty) = X \in \mathfrak{M}$.

Thus, by definition of a measurable function, χ_A is measurable iff $A \in \mathfrak{M}$. □

Definition 22 (Simple Function)

Let $(X, \mathfrak{M})$ be a measurable space. A function $f : X \to \mathbb{R}$ is called a ***simple function*** *if f is measurable and has a finite range.*

Note 11.1.2 (Standard Form of Simple Functions)

Suppose f is simple, say with the range $\{a_n\}_{n=1}^N \subseteq \mathbb{R}$. Since f is measurable, we may let

$$A_j := \{x : f(x) = a_j\} = f^{-1}(\{a_j\}) \in \mathfrak{M}.$$

We may then write

$$f(x) = \sum_{j=1}^N a_j \chi_{A_j}(x), \tag{11.1}$$

where we note that $\{A_j\}$ is a disjoint partition of X. We call Equation (11.1) the ***standard form*** *of f.*

Example 11.1.1

Consider the function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f = 2\chi_{[0,2]} + 3\chi_{[1,3]}.$$

The form of f above is not the standard form since $[0,2] \cap [1,3] = [1,2] \neq \emptyset$. We may, however, re-express f as

$$f = 2\chi_{[0,1)} + 5\chi_{[1,2]} + 3\chi_{(2,3)},$$

which is then a standard form for f.

Theorem 25 (Increasing Sequence of Simple Functions Converges an Arbitrary Measurable Function)

Let $(X, \mathfrak{M})$ be a measurable space. Let $f : X \to [0, \infty]$. Then there exists simple functions $\{\varphi_n\}_n$, such that

$$0 \leq \varphi_0 \leq \varphi_1 \leq \ldots \leq f$$

such that

$$f(x) = \lim_{n \to \infty} \varphi_n(x).$$

If f is bounded on $E \subseteq X$, then $\varphi_n \to f$ uniformly on E.

Strategy

Let's consider the bounded case. Let M be the bound on f. We construct φ_1 by subdividing $[0, M]$ into 2 equal parts, in particular considering

$$E_0 \coloneqq \left\{ x : 0 \leq f(x) \leq \frac{M}{2} \right\}$$
$$E_1 \coloneqq \left\{ x : \frac{M}{2} < f(x) \leq M \right\},$$

and letting

$$\varphi_1 = 0\chi_{E_0} + \frac{M}{2}\chi_{E_1} \leq f.$$

Note that $f(x) - \varphi_1(x) \leq \frac{M}{2}$.

Similarly, we construct φ_2 by subdividing $[0, M]$ into 4 equal parts

$$\begin{aligned} E_0 &\coloneqq \left\{ x : 0 \leq f(x) \leq \frac{M}{4} \right\} \\ E_1 &\coloneqq \left\{ x : \frac{M}{4} < f(x) \leq \frac{M}{2} \right\} \\ E_2 &\coloneqq \left\{ x : \frac{M}{2} \leq f(x) < \frac{3M}{4} \right\} \\ E_3 &\coloneqq \left\{ x : \frac{3M}{4} \leq f(x) \leq M \right\}, \end{aligned}$$

and letting

$$\varphi_2 = 0\chi_{E_0} + \frac{M}{4}\chi_{E_1} + \frac{M}{2}\chi_{E_2} + \frac{3M}{4}\chi_{E_3} \leq f.$$

Note that $f(x) - \varphi_2(x) \leq \frac{M}{4}$ and $\varphi_1 \leq \varphi_2$. We can continue doing this for $\varphi_3, \varphi_4, \ldots$, and we will show that this gives us what we want.

For the unbounded case, we can use a similar idea but consider

$$E_{n,k} \coloneqq \left\{ x : \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} \right\}$$

for $0 \leq k \leq 2^{2n} - 1$, and set

$$E_{2^{2n}} \coloneqq \left\{ x : f(x) > 2^n \right\}.$$

Then, constructing the φ_n's in a manner similar to that in the bounded case, we can prove the statement.

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12.1 Measurable Functions (Continued 3)

Proof (Proof for Theorem 25)

For $n \in \mathbb{N} \cup \{0\}$ and $0 \leq k \leq 2^{2n} - 1$, let

$$E_{n,k} := \left\{ x : \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} \right\} = f^{-1}\left[\frac{k}{2^n}, \frac{k+1}{2^n} \right),$$

and let

$$E_{n,2^{2n}} := \{ x : f(x) \geq 2^n \}.$$

Then, for each n, we define

$$\varphi_n = \sum_{k=0}^{2^{2n}} \frac{k}{2^n} \chi_{E_{n,k}},$$

which we see that each φ_n is measurable and hence a simple function. Furthermore, for each n and for all x,

$$\varphi_n(x) \leq f(x).$$

Note that for $x \in \bigcup_{k=0}^{2^{2n}-1} E_{n,k}$, we have

$$\left| f(x) - \varphi_n(x) \right| \leq \frac{1}{2^n}.$$

Thus for $f(x) \neq \infty$, $\varphi_n(x) \to f(x)$ pointwise. For $f(x) = \infty$, we must have $x \in E_{n,2^{2n}}$, which then $\varphi_n(x) = 2^n \to \infty$ as $n \to \infty$. Thus, regardless of the value of $f(x)$ for every x, we have $\varphi_n(x) \to f(x)$.

Let $\varepsilon > 0$. Now if f is bounded on some set E, say by $f(x) \leq M$

for all $x \in E$, then as soon as $2^N > M$ for some (large) $N = -\log_2 \varepsilon$, we must have that

$$x \in \bigcup_{k=0}^{2^{2n}-1} E_{n,k},$$

for each $n > N$. It follows that

$$\left|\varphi_n(x) - f(x)\right| \leq \frac{1}{2^n} < \frac{1}{2^N} = \varepsilon.$$

It follows that $\varphi_n \to f$ uniformly on E. □

12.2 *Integration of Non-Negative Functions*

Definition 23 (Integral of a Simple Function)

Let

$$\mathcal{L}^+ := \{f : X \to [0, \infty] \mid f \text{ is } \mathfrak{M}\text{-measurable}\}.$$

Given $\varphi \in \mathcal{L}^+$ *a simple function, i.e.* $\operatorname{range} \varphi = \{0 \leq a_1 < \ldots a_n\}$*, with* $E_j := \{x : \varphi(x) = a_j\}$*, the standard form for* φ *is*

$$\varphi(x) = \sum_{k=1}^{n} a_j \chi_{E_j}(x).$$

We define the ***integral*** *of* φ *as*

$$\int_X \varphi \, d\mu := \sum_{k=1}^{n} a_j \mu(E_j),$$

where we let

$$0 \cdot (\infty) = 0 = (\infty) \cdot 0.$$

If $A \in \mathfrak{M}$*, then we also define*

$$\int_A \varphi \, d\mu := \sum_{k=1}^{n} a_j \mu(E_j \cap A).$$

Proposition 26 (Properties of Integrals of Simple Functions)

Let $\varphi, \psi \in \mathcal{L}^+$ be simple functions. Then

1. *if $c \geq 0$, then $\int_X c\varphi \, d\mu = c \int_C \varphi \, d\mu$.*
2. *$\int_X (\varphi + \psi) \, d\mu = \int_X \varphi \, d\mu + \int_X \psi \, d\mu$.*
3. *$0 \leq \varphi \leq \psi \implies \int_X \varphi \, d\mu \leq \int_X \psi \, d\mu$.*
4. *Fixing φ, let*
$$\nu(A) = \int_A \varphi \, d\mu.$$
Then ν is a measure on $\mathfrak{M}$.

Proof

1. If $c = 0$, then
$$\int_X 0 \cdot \varphi \, d\mu = 0 = 0 \int_X \varphi \, d\mu.$$
If $c > 0$, then for $\varphi = \sum a_j \chi_{E_j}$,
$$c\varphi = \sum c a_j \chi_{E_j},$$
which is also a standard form. [1] Thus
$$\int_X c\varphi \, d\mu = \sum c a_j \chi_{E_j} = c \sum a_j \chi_{E_j} = c \int_X \varphi \, d\mu.$$

2. Let
$$\varphi = \sum_{j=0}^{n} a_j \chi_{E_j} \quad \text{and} \quad \psi = \sum_{i=0}^{m} b_i \chi_{F_i},$$
be the standard form for φ and ψ respectively. Note that
$$E_1 \mathbin{\dot\cup} E_2 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} E_n = X \quad \text{and} \quad F_1 \mathbin{\dot\cup} F_2 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} F_m = X.$$
Thus $\{E_j \cap F_i\}_{j=1,i=1}^{m,n}$ is a pairwise disjoint collection of X, with
$$\bigcup_{j=1}^{n} \bigcup_{i=1}^{m} E_j \cap F_i = X.$$
Now on each $E_j \cap F_i$, we have
$$\varphi + \psi = a_j + b_i.$$

[1] It is rather important that we note that this realization that $c\varphi$ is a standard form is important, since it allows us to then use Definition 23.

Thus

$$\varphi + \psi = \sum_{k=1}^{l} c_k \chi_{G_k},$$

where

$$G_k \coloneqq \{x : \varphi(x) + \psi(x) = c_k\},$$

where we note that $x \in G_k \implies x \in E_{j_0} \cap F_{i_0}$ for some j_0 and i_0, which then

$$c_k = \varphi(x) + \psi(x) = a_{j_0} + b_{i_0}.$$

It follows that

$$G_k \coloneqq \bigcup_{j,i} \{E_j \cap F_i : a_j + b_i = c_k\},$$

and so

$$\mu(G_k) = \sum_{a_j + b_i = c_k} \mu(E_j \cap F_i).$$

Thus

$$\begin{aligned} \int_X (\varphi + \psi) \, d\mu &= \sum_{k=1}^{l} c_k \mu(G_k) \\ &= \sum_{k=1}^{l} \sum_{a_j + b_i = c_k} (a_j + b_i) \mu(E_j \cap F_i) \\ &= \sum_{j=1}^{n} \sum_{i=1}^{m} (a_j + b_i) \mu(E_j \cap F_i). \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_X \varphi \, d\mu + \int_X \psi \, d\mu &= \sum_{j=1}^{n} a_j \mu(E_j) + \sum_{i=1}^{m} b_i \mu(F_i) \\ &= \sum_{j=1}^{n} a_j \sum_{i=1}^{m} \mu(E_j \cap F_i) + \sum_{i=1}^{m} b_i \sum_{j=1}^{n} \mu(E_j \cap F_i) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{m} (a_j + b_i) \mu(E_j \cap F_i). \end{aligned}$$

Hence

$$\int_X (\varphi + \psi) \, d\mu = \int_X \varphi \, d\mu + \int_X \psi \, d\mu.$$

3. Since $0 \leq \varphi \leq \psi$, the function

$$\gamma = \psi - \varphi \geq 0$$

is in $\mathcal{L}^+$, measurable and clearly simple. In particular, we notice that

$$\psi = \varphi + \gamma.$$

By (2), it follows that

$$\int_X \psi \, d\mu = \int_X \varphi \, d\mu + \int_X \gamma \, d\mu \geq \int_X \varphi \, d\mu.$$

4. Fix

$$\varphi = \sum_{j=1}^{n} a_j \chi_{E_j}.$$

Then

$$\nu(A) = \int_A \varphi \, d\mu = \sum_{j=1}^{n} a_j \mu(E_j \cap A).$$

Showing that ν is a measure on $\mathfrak{M}$

- For $A = \emptyset$, we have

$$\nu(\emptyset) = \sum_{j=1}^{n} a_j \mu(E_j \cap \emptyset) = \sum_{j=1}^{n} a_j \mu(\emptyset) = 0.$$

- Let $A = \dot{\bigcup}_{i=1}^{N} A_i$. Then

$$\begin{aligned} \nu(A) &= \sum_{j=1}^{n} a_j \mu(E_j \cap A) = \sum_{j=1}^{n} a_j \left[\sum_{i=1}^{N} \mu(E_j \cap A_i) \right] \\ &= \sum_{i=1}^{N} \sum_{j=1}^{n} a_j \mu(E_j \cap A_i) = \sum_{i=1}^{N} \int_{A_i} \varphi \, d\mu \\ &= \sum_{i=1}^{N} \nu(A_i). \end{aligned}$$

□

Corollary 27 (Integral of Simple Functions not in Standard Form)

Suppose $\varphi \in \mathcal{L}^+$ *is simple. Suppose we express*

$$\varphi = \sum_{j=1}^{n} b_j \chi_{F_j},$$

not necessarily in standard form, where $b_j \geq 0$. *Then*

$$\int_X \varphi \, d\mu = \sum_{j=1}^{n} b_j \mu(F_j).$$

Proof

The proof is similar to that of (2) in Proposition 26. □

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13.1 Integration

We are now ready to define the integration for an arbitrary measurable function.

Definition 24 (Integral of a Measurable Function)

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Suppose $f \in \mathcal{L}^+$. We define

$$\int_X f \, d\mu := \sup \left\{ \int_X \varphi \, d\mu : 0 \leq \varphi \leq f, \, \varphi \text{ is simple} \right\}.$$

Remark 13.1.1

Notice that for a simple function φ, we now have seemingly 2 definitions for its integral. However, it is not difficult to realize that the 2 definitions agree. In particular, φ itself is one of the simple functions in the set of which we take the supremum, and in particular φ itself is the supremum.

Proposition 28 (Properties of the Integral of Measurable Functions)

Let $(X, \mathfrak{M}, \mu)$ be a measure space, and $f \in \mathcal{L}^+$.

1. *If $c \geq 0$, then*

$$\int_X cf \, d\mu = c \int_X f \, d\mu.$$

2. *If $g \in \mathcal{L}^+$ such that $0 \leq g \leq f$, then*

$$\int_X g \, d\mu \leq \int_X f \, d\mu.$$

Proof

1. If $c = 0$, then

$$\int_X 0 \cdot f \, d\mu = 0 = 0 \int_X f \, d\mu.$$

If $c > 0$, then

$$\int_X cf \, d\mu = \sup \left\{ \int_X \varphi \, d\mu : 0 \leq \leq \varphi \leq cf, \, \varphi \text{ is simple} \right\}.$$

But $\varphi \leq cf \iff c^{-1}\varphi \leq f$, and $\psi := c^{-1}\varphi$ is simple. In particular, we can thus have

$$c\psi = \varphi \leq cf.$$

Thus

$$\begin{aligned} \int_X cf \, d\mu &= \sup \left\{ \int_X \varphi \, d\mu : 0 \leq \varphi \leq cf, \, \varphi \text{ is simple} \right\} \\ &= \sup \left\{ \int_X c\psi \, d\mu : 0 \leq c\psi \leq cf, \, \psi \text{ is simple} \right\} \\ &= \sup \left\{ c \int_X \psi \, d\mu : 0 \leq \psi \leq f, \, \psi \text{ is simple} \right\} \\ &= c \sup \left\{ \int_X \psi \, d\mu : 0 \leq \psi \leq f, \, \psi \text{ is simple} \right\} \\ &= c \cdot \int_X f \, d\mu \end{aligned}$$

2. Notice that

$$\{\psi : 0 \leq \psi \leq g, \, \psi \text{ is simple} \} \subseteq \{\varphi : 0 \leq \varphi \leq f, \, \varphi \text{ is simple} \}.$$

Thus

$$\begin{aligned} \int_X g \, d\mu &= \sup \left\{ \int_X \psi \, d\mu : 0 \leq \psi \leq g, \, \psi \text{ is simple} \right\} \\ &\leq \sup \left\{ \int_X \varphi \, d\mu : 0 \leq \varphi \leq f, \, \varphi \text{ is simple} \right\} = \int_X f \, d\mu \quad \square \end{aligned}$$

Example 13.1.1

Consider $f_n = n \cdot \chi_{(0, \frac{1}{n})}$. We see that $\forall x \in \mathbb{R}$,

$$\lim_{n \to \infty} f_n(x) = 0,$$

i.e. that $f_n \to f \equiv 0$ pointwise. However, notice that under the Lebesgue measure

$$\int_{\mathbb{R}} f_n \, d\mu = n \cdot \frac{1}{n} = 1,$$

for each n. Thus

$$\lim_{n \to \infty} \int_{\mathbb{R}} f_n \, d\mu = 1 \neq 0 = \int_{\mathbb{R}} \lim_{n \to \infty} f_n \, d\mu.$$

Warning

The above example shows that the limit of the integral of a sequence of measurable functions need not be the integral of the limit of the sequence of measurable functions, i.e. it ***need not be the case*** *that*

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X \lim_{n \to \infty} f_n \, d\mu.$$

In other words, limits do not behave nicely with our definition of integration for arbitrary measurable functions.

We shall see that not all hope is loss, and there are indeed some sequences of functions which have this desirable property.

Theorem 29 (Monotone Convergence Theorem (MCT))

Suppose $\{f_n\}_n \subseteq \mathcal{L}^+$ *such that* $f_n \leq f_{n+1}$. *Let*

$$f(x) = \lim_{n \to \infty} f_n(x) = \sup_{n \geq 1} f_n(x)$$

for all $x \in X$. *Then*

$$\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu = \sup_{n \geq 1} \int_X f_n \, d\mu.$$

Proof

By (2) of Proposition 28, we have that

$$\int_X f_n \, d\mu \leq \int_X f_{n+1} \, d\mu.$$

Thus we indeed have

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \sup_{n \geq 1} \int_X f_n \, d\mu.$$

It is also easy to see that since $f_n \leq f$ for each n, we have, by the same reasoning as above

$$\int_X f_n \, d\mu \leq \int_X f \, d\mu$$

for each n, and so

$$\sup_{n \geq 1} \int_X f_n \, d\mu \leq \int_X f \, d\mu.$$

It remains to show that

$$\sup_{n \geq 1} \int_X f_n \, d\mu \geq \int_X f \, d\mu.$$

To do this, fix an α such that $0 < \alpha < 1$. Consider a simple function φ such that $0 \leq \varphi \leq f$. Then $\forall x \in X$, $\varphi(x) \leq f(x)$. Now let

$$E_n \coloneqq \{x : f_n(x) \geq \alpha\varphi(x)\}.$$

Notice that not all $E_n = \emptyset$, since φ is fixed and $\lim_{n \to \infty} f_n(x) = f(x)$ by assumption. Observe that

$$\bigcup_n E_n = X$$

and

$$E_1 \subseteq E_2 \subseteq \ldots.$$

Recall that (4) of Proposition 26 tells us that

$$\nu(E) := \int_E \alpha\varphi \, d\mu$$

is a measure, and we know that measures are continuous from below. It follows that

$$\int_X \alpha\varphi \, d\mu = \nu(X) = \lim_{n\to\infty} \nu(E_n) = \lim_{n\to\infty} \int_{E_n} \alpha\varphi \, d\mu.$$

Now on each E_n, we know that $f_n \geq \alpha\varphi$. Thus

$$\int_X f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq \int_{E_n} \alpha\varphi \, d\mu,$$

for each n, which then

$$\lim_{n\to\infty} \int_X f_n \, d\mu \geq \lim_{n\to\infty} \int_{E_n} f_n \, d\mu = \int_X \alpha\varphi \, d\mu.$$

Hence

$$\sup_{n\geq 1} \int_X f_n \, d\mu \geq \int_X \alpha\varphi \, d\mu$$

for every simple function $\varphi \leq f$. This implies that

$$\begin{aligned} \sup_{n\geq 1} \int_X f_n \, d\mu &\geq \alpha \sup \left\{ \int_X \varphi \, d\mu : 0 \leq \varphi \leq f, \, \varphi \text{ is simple} \right\} \\ &= \alpha \int_X f \, d\mu. \end{aligned}$$

Since $\sup\{\alpha : 0 < \alpha < 1\} = 1$, it follows that

$$\sup_{n\geq 1} \int_X f_n \, d\mu \geq \int_X f \, d\mu. \qquad \square$$

Corollary 30 (Addition of Integrals of Measurable Functions)

Let $f, g \in \mathcal{L}^+$. Then

$$\int_X (f+g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu.$$

Proof

By Theorem 25, $\exists \{\varphi_n\}_n, \{\psi_n\}_n$ such that

$$\varphi_n \nearrow f \quad \text{and} \quad \psi_n \nearrow g$$

both pointwise. Clearly then

$$\varphi_n + \psi_n \nearrow f + g.$$

By the MCT, we have

$$\begin{aligned} \int_X (f+g)\,d\mu &\overset{MCT}{=} \sup_{n\geq 1} \int_X (\varphi_n + \psi_n)\,d\mu \\ &= \sup_{n\geq 1} \left(\int_X \varphi_n\,d\mu + \int_X \psi_n\,d\mu \right) \\ &= \sup_{n\geq 1} \int_X \varphi_n\,d\mu + \sup_{n\geq 1} \int_X \psi_n\,d\mu \\ &\overset{MCT}{=} \int_X f\,d\mu + \int_X g\,d\mu. \end{aligned}$$

□

Corollary 31 (Interchanging Infinite Sums and the Integral Sign)

Let $\{f_n\}_n \subseteq \mathcal{L}^+$ and

$$s(x) = \sum_{n=1}^{\infty} f_n(x).$$

Then

$$\int_X s\,d\mu = \sum_{n=1}^{\infty} \int_X f_n\,d\mu,$$

i.e.

$$\int_X \sum_{n=1}^{\infty} f_n\,d\mu = \sum_{n=1}^{\infty} \int_X f_n\,d\mu.$$

Proof

For each $N \in \mathbb{N} \setminus \{0\}$, let

$$s_N(x) = \sum_{n=1}^{N} f_n(x).$$

Then since $f_n \geq 0$, we have that $s_N \nearrow s$. By the MCT, we have

$$\begin{aligned}\int_X s\, d\mu &\overset{MCT}{=} \lim_{N\to\infty} \int_X s_N \, d\mu \\ &= \lim_{N\to\infty} \int_X \sum_{n=1}^{N} f_n \, d\mu \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \int_X f_n \, d\mu \\ &= \sum_{n=1}^{\infty} \int_X f_n \, d\mu. \end{aligned}$$

□

Definition 25 (Almost Everywhere)

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Let $E \in \mathfrak{M}$. Let (P) be a property. We say P holds ***almost everywhere (a.e.)*** *if the set*

$$B = \{x \in E : (P) \text{ does not hold for } x\}$$

has measure zero, i.e. $\mu(B) = 0$.

Example 13.1.2

We say that $f = 0$ a.e. iff

$$\mu(\{x : f(x) \neq 0\}) = 0.$$

Proposition 32 (Almost Everywhere Zero Functions have Zero Integral)

If $f \in \mathcal{L}^+$, then

$$\int_X f\, d\mu = 0 \iff f = 0 \text{ a.e.}$$

Proof

($\implies$) Observe that our supposition says that

$$0 = \int_X f \, d\mu = \sup \left\{ \int_X \varphi \, d\mu : 0 \leq \varphi \leq f, \, \varphi \text{ is simple} \right\}.$$

In particular, this means that $\forall \varphi \leq f$ simple, we have that $\int_X \varphi \, d\mu = 0$. Notice if we write the simple function φ as its standard form, i.e.

$$\varphi = \sum_{n=1}^{N} a_n \chi_{E_n} \quad a_n \geq 0,$$

then

$$0 = \int_X \varphi \, d\mu = \sum_{n=1}^{N} a_n \mu(E_n).$$

Then if $a_n > 0$, we must have $\mu(E_n) = 0$. On the other hand, if $a_n = 0$, then $\mu(E_n)$ can be anything but it will not contribute to the sum. In other words,

$$\mu(\{x : \varphi(x) \neq 0\}) = 0,$$

and so $\varphi = 0$ a.e.

Consider $\varphi_n \nearrow f$ pointwise, which we can get from Theorem 25. By the above argument, for each n, the set

$$B_n \coloneqq \{x : \varphi_n(x) \neq 0\}$$

has measure zero, i.e. $\mu(B_n) = 0$. Let $B = \bigcup_{n=1}^{\infty} B_n$. Then by subadditivity,

$$\mu(B) = 0.$$

For each $x \notin B$, we have that $\varphi_n(x) = 0$, for every n. Since $\varphi_n \nearrow f$, we have that

$$\forall x \notin B \quad f(x) = 0.$$

Thus

$$\begin{aligned} B^C &\subseteq \{x : f(x) = 0\} \\ \implies B &\supseteq \{x : f(x) \neq \emptyset\} \\ \implies \mu(\{x &: f(x) \neq 0\}) \leq \mu(B) = 0 \end{aligned}$$

$$\implies \mu(\{x : f(x) \neq 0\}) = 0.$$

($\impliedby$) Since $f = 0$ a.e., we have

$$\mu(\{x : f(x) \neq 0\}) = 0.$$

Let $\varphi_n \nearrow f$ by Theorem 25. Then $\varphi_n = 0$ a.e. Let

$$\varphi_n = \sum_j a_{n,j} \chi_{E_{n,j}}$$

be the standard form of φ_n for each n. Then if $a_{n,j} \neq 0$, we must have $\mu(E_{n,j}) = 0$. This implies that

$$\int_X \varphi_n \, d\mu = \sum_j a_{n,j} \mu(E_{n,j}) = 0$$

for all n . By the MCT, we have

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X \varphi_n \, d\mu = 0. \qquad \square$$

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Homework (Homework 10)

Suppose $f : \mathbb{R} \to \mathbb{R}$ is increasing. Prove that f is $\mathfrak{B}(\mathbb{R})$-measurable.

Homework (Homework 11)

Suppose $(X, \mathfrak{M}, \mu)$ is a measure space, and $f \in \mathcal{L}^+$. Let $\{E_n\}_n \subseteq \mathfrak{M}$ be a pairwise disjoint set, and $E = \bigcup_n E_n$. Prove that

$$\int_E f \, d\mu = \sum_{n=1}^{\infty} \int_{E_n} f \, d\mu.$$

Homework (Homework 12)

Let $f : [0,1] \to \mathbb{R}_e$ be Lebesgue measurable, $f \geq 0$, and $\int_{[0,1]} f \, d\mu < \infty$. Prove that

$$\int_{[0,1]} x^k f(x) \, d\mu \to 0 \text{ as } k \to \infty.$$

What if $\int_{[0,1]} f \, d\mu = \infty$? Prove that it is still true or give a counter example.

Homework (Homework 13)

Suppose $F : \mathbb{R} \to \mathbb{R}$ is increasing and right continuous. Suppose $E \in \mathfrak{M}$

with $\mu_F(E) < \infty$. Given $\varepsilon > 0$, prove that there exists a set

$$A = \bigcup_{k=1}^{\infty} (a_k, b_k]$$

such that

$$\mu_F((E \setminus A) \cup (A \setminus E)) < \varepsilon.$$

*(Note: $E \triangle A := (E \setminus A) \cup (A \setminus E)$ is known as the **symmetric difference** of E and A.)*

14.1 *Integration (Continued)*

Notation

We have used a similar notation earlier on for a sequence of values. We shall use the same notation for a sequence of functions.

By $f_n \nearrow f$ a.e., we mean $\exists B$ a set such that $\mu(B) = 0$, such that $\forall x \notin B$, we have

$$f_1(x) \leq f_2(x) \leq \ldots \leq f(x)$$

and

$$\lim_{n \to \infty} f_n(x) = f(x).$$

Corollary 33 (Monotone Convergence Theorem (A.E. Ver.))

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Suppose $\{f_n\}_n \subseteq \mathcal{L}^+$ such that $f_n \nearrow f$ a.e. Then

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Proof

Let B be a set such that $\mu(B) = 0$, such that $\forall x \notin B$

$$f_1(x) \leq f_2(x) \leq \ldots \leq f(x)$$

and

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Then we may write

$$f = f \cdot \chi_{B^C} + f \cdot \chi_B.$$

By Corollary 30, we have

$$\int_X f \, d\mu = \int_X f \cdot \chi_{B^C} \, d\mu + \int_X f \cdot \chi_B \, d\mu = \int_X f \cdot \chi_{B^C} \, d\mu + 0.$$

Observe that on B^C, we have $f_n \cdot \chi_{B^C} \nearrow f \cdot \chi_B$. Thus by the Monotone Convergence Theorem (MCT)

$$\int_X f \, d\mu = \int_X f \cdot \chi_{B^C} \, d\mu = \lim_{n\to\infty} \int_X f_n \cdot \chi_{B^C} \, d\mu.$$

Finally, observe that

$$\int_X f_n \, d\mu = \int_X f_n \cdot \chi_{B^C} \, d\mu + \int_X f_n \cdot \chi_B \, d\mu = \int_X f_n \cdot \chi_{B^C} \, d\mu + 0.$$

Thus indeed

$$\int_X f \, d\mu = \lim_{n\to\infty} f_n \, d\mu.$$

□

Theorem 34 (Fatou's Lemma)

Let $\{f_n\}_n \subseteq \mathcal{L}^+$, and

$$f(x) = \liminf_{n\geq 1} f_n(x)$$

is measurable. Then

$$\int_X f \, d\mu = \int_X \liminf_{n\geq 1} f_n \, d\mu \leq \liminf_{n\geq 1} \int_X f_n \, d\mu.$$

Proof

Let

$$g_k(x) = \inf_{n\geq k} f_n(x).$$

Then $g_k \nearrow f(x)$. By ★ Monotone Convergence Theorem (MCT),

$$\int_X f \, d\mu = \lim_{k \to \infty} \int_X g_k \, d\mu = \sup_{k \geq 1} \int_X g_k \, d\mu.$$

Notice that $\forall n \geq k$, by construction, $g_k \leq f_n$. Thus (by Proposition 28), $\forall n \geq k$,

$$\int_X g_k \, d\mu \leq \int_X f_n \, d\mu.$$

This implies that

$$\int_X g_k \, d\mu \leq \inf_{n \geq k} \int_X f_n \, d\mu.$$

It follows that

$$\int_X f \, d\mu = \sup_{k \geq 1} \int_X g_k \, d\mu \leq \sup_{k \geq 1} \inf_{n \geq k} \int_X f_n \, d\mu = \liminf_{n \geq 1} \int_X f_n \, d\mu. \quad \square$$

Example 14.1.1

Recall our example where

$$f_n = n \cdot \chi_{(0, \frac{1}{n})}$$

and

$$\lim_{n \to \infty} f_n(x) = f(x) = 0.$$

We see that

$$\int_X f \, d\mu = 0 < 1 = \liminf_{n \geq 1} \int_X f_n \, d\mu.$$

Therefore, we do not always expect an equality to happen vis-a-vis ★ Fatou's Lemma.

The following propositions have pretentious names, but we will see why right after looking at them.

Proposition 35 (Integrable Functions have Value at Infinity over a Set of Measure Zero)

Let $f \in \mathcal{L}^+$ and $\int_X f \, d\mu < \infty$. Then

$$\mu(\{x : f(x) = \infty\}) = 0.$$

Proof

Let $A = \{x : f(x) = \infty\}$. Consider the sequence of simple functions

$$\varphi_n = n \cdot \chi_A.$$

Then when not on A, we have $\varphi_n = 0 \leq f$, and on A, we have $\varphi_n = n < \infty = f$. Thus $\varphi_n \leq f$. Therefore, $\forall n \geq 1$

$$n\mu(A) = \int_X \varphi_n \, d\mu \leq \int_X f \, d\mu < \infty.$$

In particular, $\forall n \geq 1$,

$$\mu(A) = \frac{1}{n} \int_X f \, d\mu < \infty.$$

It follows that

$$\mu(A) = 0.$$

□

Proposition 36 (Set where the Integrable Function is Strictly Positive is σ-finite)

*Let $f \in \mathcal{L}^+$ and $\int_X f \, d\mu < \infty$. Then $\{x : f(x) > 0\}$ is σ-**finite**, [1] i.e. it is expressible as a union of subsets which have finite measure.*

[1] This σ-finite has a similar meaning to the σ-finite we have seen before.

Proof

Let $E = \{x : f(x) > 0\}$. Consider $E_n = \left\{x : f(x) \geq \frac{1}{n}\right\}$. Then $E = \bigcup_{n=1}^{\infty} E_n$.

For each $n \geq 1$, let $\varphi_n = \frac{1}{n}\chi_{E_n} \leq f$. Then

$$\frac{1}{n}\mu(E_n) = \int_X \varphi_n \, d\mu \leq \int_X f \, d\mu < \infty.$$

Hence

$$\mu(E_n) \leq \frac{1}{n} \int_X f \, d\mu < \infty.$$

It follows that, indeed, each E_n has finite measure.

□

14.2 Integration of Real- and Complex-Valued Functions

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Let $f : X \to \mathbb{R}_e$ be measurable. If we write $f = f^+ - f^-$, recall that f^+ and f^- are both measurable. Furthermore, $\left|f\right| = f^+ + f^-$. We observe that

$$\int_X f^+ \, d\mu \leq \int_X \left|f\right| d\mu = \int_X f^+ \, d\mu + \int_X f^- \, d\mu,$$

and similarly for f^-. Thus

$$\int_X \left|f\right| d\mu < \infty \iff \int_X f^+ \, d\mu, \int_X f^- \, d\mu < \infty.$$

Definition 26 (Integrable Function)

Let $(X, \mathfrak{M}, \mu)$ be an arbitrary measure space. Let $f : X \to \mathbb{R}_e$ be a measurable function. We say that f is ***integrable*** *if*

$$\int_X f^+ \, d\mu < \infty \quad \text{and} \quad \int_X f^- \, d\mu < \infty.$$

Remark 14.2.1

Since $f = f^+ - f^-$, by Corollary 30, we have

$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu.$$

Remark 14.2.2 (On complex functions)

Consider $f : X \to \mathbb{C} \simeq \mathbb{R}^2$. We know that $\forall z = a + ib \in \mathbb{C}$, we may write $z = (a, b) \in \mathbb{R}^2$, with

$$\mathfrak{R}(z) = a \quad \text{and} \quad \mathfrak{I}(z) = b.$$

Notice that

$$|z| = \sqrt{\bar{z}z} = \sqrt{a^2 + b^2}.$$

Thus

$$|a| \leq |z| \quad \text{and} \quad |b| \leq |z|.$$

We observe that

$$|z| = |a + ib| \leq |a| + |b| = 2|z|.$$

Now if we let

$$f = \Re(f) + \Im(f),$$

then by a similar line of thought as above,

$$\begin{aligned} |f| &\leq |\Re(f)| + |\Im(f)| \\ &= \Re(f)^+ + \Re(f)^- + \Im(f)^+ + \Im(f)^- \\ &\leq 4|f|. \end{aligned}$$

Then by the same argument that we've seen at the beginning of this section,

$$\int_X |f| \, d\mu$$

$$\iff$$

$$\int_X \Re(f)^+ \, d\mu, \int_X \Re(f)^- \, d\mu, \int_X \Im(f)^+ \, d\mu, \int_X \Im(f)^- \, d\mu < \infty.$$

Therefore, we say that $f : X \to \mathbb{C}$ is integrable if all the above 4 integrals are finite. In particular, we can set

$$\begin{aligned} \int_X f \, d\mu &= \int_X \Re(f) \, d\mu + i \int_X \Im(f) \, d\mu \\ &= \left[\int_X \Re(f)^+ \, d\mu - \int_X \Re(f)^- \, d\mu \right] + i \left[\int_X \Im(f)^+ \, d\mu - \int_X \Im(f)^- \, d\mu \right]. \end{aligned}$$

This shows that it suffices for us to focus on studying real-valued functions to understand complex-valued functions within our context.

Notation

Let $(X, \mathfrak{M}, \mu)$ be a measure space. We write

$$\mathcal{L}^1 := \{ f : X \to \mathbb{R}_e \mid \int_X |f| \, d\mu < \infty \},$$

and

$$\mathcal{L}^1_{\mathbb{C}} := \{f : X \to \mathbb{C} \mid \int_X |f| \, d\mu < \infty\}.$$

Proposition 37 ($\mathcal{L}^1$ is a Vector Space)

$\mathcal{L}^1$ is a vector space. Furthermore, for $f, g \in \mathcal{L}^1$, we have

1. $\int_X f + g \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$.
2. $\forall a \in \mathbb{R} \quad \int_X af \, d\mu = a \int_X f \, d\mu$.

Note that this is also true for $\mathcal{L}^1_{\mathbb{C}}$.

Proof

$\mathcal{L}^1$ is a vector space Let $f, g \in \mathcal{L}^1$. Since $|f+g| \leq |f| + |g|$, we have that

$$\int_X |f+g| \, d\mu \leq \int_X |f| \, d\mu + \int_X |g| \, d\mu < \infty.$$

Thus $f + g \in \mathcal{L}^1$.

For any $a \in \mathbb{R}$, $|af| = |a| \, |f|$, and so

$$\int_X |af| \, d\mu = |a| \int_X f \, d\mu < \infty.$$

Thus $af \in \mathcal{L}^1$.

Linearity in $\mathcal{L}^1$ Let $h = h^+ - h^- = f + g = f^+ - f^- + g^+ - g^-$. Note $f = f^+ - f^-$ and $g = g^+ - g^-$. WTS

$$\int_X h \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu + \int_X g^+ \, d\mu - \int_X g^- \, d\mu.$$

By rearrangement, we have that

$$h^+ + f^- + g^- = h^- + f^+ + g^+,$$

where we rearrange them so that all the functions are now non-negative. By Corollary 30, we have [2]

[2] For sanity, let's drop the $d\mu$ and subscript X here. We shall do this when the context is clear.

$$\int h^+ + \int f^- + \int g^- = \int h^- + \int f^+ + \int g^+.$$

It follows that, indeed,

$$\int h^+ - \int h^- = \int f^+ - \int f^- + \int g^+ - \int g^-.$$

Now for any $a \in \mathbb{R}$,

$$af = af^+ - af^-.$$

Since

$$\int af = \int af^+ - \int af^-,$$

and each of the functions on the RHS is non-negative, by Proposition 28,

$$\int af = a\int f^+ - a\int f^- = a\int f. \qquad \square$$

Proposition 38 (Absolute Value of Integral is Lesser Than Integral of Absolute Value)

If $f \in \mathcal{L}^1$, then

$$\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu.$$

Proof

Since $f = f^+ - f^-$, we have

$$\left| \int f \right| = \left| \int f^+ - \int f^- \right| \leq \int f^+ + \int f^- = \int |f|. \qquad \square$$

Proposition 39 (Sub-properties of $\mathcal{L}^1$ functions)

1. *If $f \in \mathcal{L}^1$, then*

   $$\{x : f(x) \neq 0\}$$

   is σ-finite.

2. *If $f, g \in \mathcal{L}^1$, then*

$$\begin{aligned} &\forall E \in \mathfrak{M} \quad \int_E f \, d\mu = \int_E g \, d\mu \\ &\iff \int_X \left| f - g \right| d\mu = 0 \\ &\iff f = g \text{ a.e.} \end{aligned}$$

Exercise 14.2.1

The proof of (1) in Proposition 39 is left as an easy exercise of which the reader may refer to an earlier proof for reference.

We shall prove (2) in the next lecture.

15 Lecture 15 Oct 7th 2019

15.1 Integration of Real- and Complex-Valued Functions (Continued)

Proof (of (2) in Proposition 39)

(2) $\iff$ (3) We have that

$$\int_X |f-g|\,d\mu = 0 \iff |f-g| = 0 \text{ a.e.} \iff f = g \text{ a.e.}$$

(3) $\implies$ (1) $f = g$ a.e. means that $\int_E f\,d\mu = \int_E g\,d\mu$ for any $E \in \mathfrak{M}$.

(1) $\implies$ (3) Since $\int_E f = \int_E g$ for all $E \in \mathfrak{M}$, in particular, we have that on

$$E = \{x : f(x) - g(x) > 0\} \in \mathfrak{M},$$

we have

$$\int_E (f-g)\,d\mu = 0.$$

This means that $f = g$ a.e. on E, i.e.

$$\mu(\{x \in E : f(x) - g(x) > 0\}) = 0.$$

Let $\mathcal{E} = \{x \in E : f(x) - g(x) > 0\}$. Similarly, on

$$F = \{x : f(x) - g(x) < 0\} \in \mathfrak{M},$$

we have

$$\mu(\{x \in F : g(x) - f(x) > 0\}) = 0.$$

Let $\mathcal{F} = \{x \in F : g(x) - f(x) > 0\}$. It follows that

$$\mu(\{x \in X : f(x) \neq g(x)\}) \leq \mu(\mathcal{E}) + \mu(\mathcal{F}) = 0,$$

i.e. that $f = g$ a.e. on X. □

Theorem 40 (Lebesgue's Dominated Convergence Theorem)

Let $(X, \mathfrak{M}, \mu)$ be a measure space, $\{f_n\}_n \subseteq \mathcal{L}^1$ be a sequence of measurable functions such that $f_n \to f$ pointwise a.e., where f is also measurable. Suppose $\exists g \in \mathcal{L}^1$ such that

$$\forall n \geq 1 \quad |f_n(x)| \leq g(x) \text{ a.e.}$$

Then $f \in \mathcal{L}^1$ and

$$\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu.$$

Proof

For each $n \geq 1$, we have $|f_n(x)| \leq g(x)$ a.e.. In particular, $f_n(x) \leq g(x)$ a.e.. Thus $f(x) \leq g(x)$ a.e. Also, $-f_n(x) \leq g(x)$ a.e. and so $-f(x) \leq g(x)$ a.e.. Thus $|f(x)| \leq g(x)$ a.e., and so

$$\int_X |f| \, d\mu \leq \int_X g \, d\mu < \infty.$$

Thus $f \in \mathcal{L}^1$.

Now, since $f_n \to f$ pointwise a.e., we also have that $g + f_n \to g + f$ pointwise a.e. In particular, by Fatou's Lemma,

$$\int g + \int f = \int (g + f) \leq \liminf_n \int (g + f_n) = \int g + \liminf_n \int f_n.$$

Thus

$$\int f \leq \liminf_n \int f_n.$$

Similarly, we have $g - f_n \to g - f$ pointwise a.e., and so

$$\begin{aligned} \int g - \int f = \int (g - f) \leq \liminf_n \int (g - f_n) \\ &= \int g + \liminf_n \left(- \int f_n \right) \\ &= \int g - \limsup_n \int f_n. \end{aligned}$$

Thus

$$\int f \geq \limsup_n \int f_n.$$

It follows that

$$\limsup_n \int f_n \leq \int f \leq \liminf_n \int f_n \leq \limsup_n f_n.$$

Thus

$$\int f = \limsup_n \int f_n = \liminf_n \int f_n,$$

which implies that the limit exists, and so

$$\int f = \lim_n \int f_n,$$

as desired. □

Corollary 41 (A Series Convergence Test for Integrable Functions)

Let $\{f_n\}_n \subseteq \mathcal{L}^1$, and suppose

$$\sum_{n=1}^{\infty} \left(\int_X |f_n| \, d\mu \right) < \infty.$$

Then $\sum_{n=1}^{\infty} f_n$ converges a.e.

If we let $f(x) = \sum_{n=1}^{\infty} f_n(x)$ a.e., then

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

Proof

Let $g(x) = \sum_{n=1}^{\infty} \left| f_n(x) \right|$.[1] For $N \geq 1$, let

$$s_N(x) = \sum_{n=1}^{N} \left| f_n(x) \right|.$$

Then $s_N \nearrow g$. By the Monotone Convergence Theorem (MCT),

$$\int g = \lim_N \int s_N = \lim_N \sum_{n=1}^{N} \int \left| f_n \right| = \sum_{n=1}^{\infty} \int \left| f_n \right| < \infty.$$

Thus $g \in \mathcal{L}^1$. By a similar reasoning to Proposition 35, if we let $\mathcal{N} := \{x : g(x) = \infty\}$, then $\mu(\mathcal{N}) = 0$. Then for $x \notin \mathcal{N}$, we have that $g(x) = \sum_{n=1}^{\infty} \left| f_n(x) \right| < \infty$. Thus $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely on $\mathcal{N}^C$. This implies that $\sum_{n=1}^{\infty} f_n(x)$ converges a.e.

Now set

$$f(x) = \begin{cases} \sum_{n=1}^{\infty} f_n(x) & x \notin \mathcal{N} \\ 0 & x \in \mathcal{N} \end{cases}.$$

[2] Let $h_N(x) = \sum_{n=1}^{N} f_n(x)$ for each $N \geq 1$. Then $h_N \to f$ pointwise a.e.. Observe that $|h_N| \leq \sum_{n=1}^{N} \left| f_n \right| \leq g$ a.e.. By the Lebesgue's Dominated Convergence Theorem, we have that

$$\int f = \lim_N \int h_N = \lim_N \sum_{n=1}^{N} \int f_n = \sum_{n=1}^{\infty} \int f_n,$$

as desired. □

[1] It could be that $g(x) = \infty$ for some x.

[2] We can set $f(x)$ to be anything for $x \in \mathcal{N}$.

Example 15.1.1

Consider the function

$$f(x) = \begin{cases} x^{-\frac{1}{2}} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

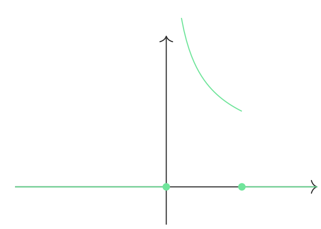

Figure 15.1: Graph of f in Example 15.1.1

From PMATH450, since f is bounded on $(0, 1)$, Lebesgue's integral coincides with Riemann's integral, and so

$$\int f\,d\mu = \int_0^1 \frac{1}{\sqrt{x}}\,dx = 2.$$

Now, let $\{r_n\}_n$ be an enumeration of $\mathbb{Q}$. Let

$$g(x) := \sum_{n=1}^{\infty} \frac{1}{2^n} f(x - r_n).$$

Since $g(x) \geq 0$, by the Monotone Convergence Theorem (MCT),

$$\int g = \sum_{n=1}^{\infty} \int \frac{1}{2^n} f(x - r_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot 2 < \infty.$$

Thus g is integrable. However, g is unbounded on every open interval, and in particular, it is discontinuous at every rational point, with

$$\lim_{x \to r_n^+} g(x) = \infty.$$

Theorem 42 (Littlewood's Second Principle, for a general measure)

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Let $f \in \mathcal{L}^1$ and $\varepsilon > 0$. Then there exists a simple function $\varphi \in \mathcal{L}^1$ such that

$$\int_X (f - \varphi)\,d\mu < \varepsilon.$$

If $\mu = \mu_F$ is a Lebesgue-Stieltjes measure on $\mathbb{R}$, then there exists a function g that vanishes outside of a bounded interval such that

$$\int_X \left| f - g \right| d\mu_F < \varepsilon.$$

Proof

Let $f = f^+ - f^-$. By Theorem 25, $\exists \varphi_n \nearrow f^+$ and $\exists \psi_n \nearrow f^-$. By the ★ Monotone Convergence Theorem (MCT), we have

$$\int f^+ = \lim_n \int \varphi_n \quad \text{and} \quad \int f^- = \lim_n \int \psi_n.$$

In particular, we have

$$\int (f^+ - \varphi_n) \to 0 \quad \text{and} \quad \int (f^- - \psi_n) \to 0.$$

This means that

$$\begin{aligned} \exists N_1 \;\; \forall n \geq N_1 \;\; \int (f^+ - \varphi_n) < \frac{\varepsilon}{2} \\ \exists N_2 \;\; \forall n \geq N_2 \;\; \int (f^- - \psi_n) < \frac{\varepsilon}{2}. \end{aligned}$$

Picking $N = \max\{N_1, N_2\}$, we have that $\forall n \geq N$,

$$\int \left| f - (\varphi_1 - \psi_n) \right| \leq \int \left| f^+ - \varphi_n \right| + \int \left| f^- - \psi_n \right| < \varepsilon.$$

This completes the first part.

Now suppose $\mu = \mu_F$. By the last part,

$$\varphi = \sum_{n=1}^{N} a_n \chi_{E_n}$$

be a simple function such that $\int \left| f - \varphi \right| d\mu_F < \varepsilon$. By Homework , for each E_n, let

$$A_n = \bigcup_{j=1}^{\infty} (a_j, b_j],$$

such that

$$\mu_F(E_n \triangle A_n) < \frac{\varepsilon}{|a_n| \, N}.$$

[3] Consider the simple function

$$\psi = \sum_n a_n \chi_{A_n}.$$

Then

$$\begin{aligned} \int \left| \varphi - \psi \right| d\mu_F &\leq \sum_{n=1}^{N} |a_n| \int |\chi_{E_n} - \chi_{A_n}| \, d\mu_F \\ &= \sum_{n=1}^{N} |a_n| \, \mu(E_n \triangle A_n) < \varepsilon. \end{aligned}$$

[3] This is so that we can get a more well-understood set, and intervals are quite well-understood and easy to grasp.

It follows that

$$\int |f - \psi| \, d\mu_F \leq \int |f - \varphi| \, d\mu_F + \int |\varphi - \psi| \, d\mu_F < 2\varepsilon. \qquad \square$$

The details for the rest of this proof shall be left to the reader as an exercise. Refer to Figure 15.2.

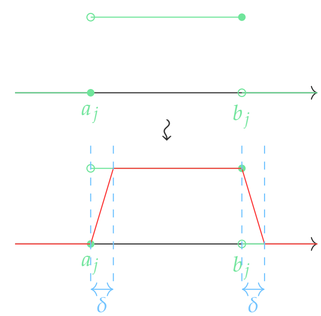

Figure 15.2: Idea for constructing the continuous function

16 Lecture 16 Oct 9th 2019

16.1 Riemann Integration VS Lebesgue Integration

We shall take a quick detour into the agreement between Riemann integration and Lebesgue integration on bounded functions. For further details, you may wish to refer to notes on PMATH450. However, in this section, we shall look at using so-called upper and lower envelopes of a function to prove the same result.

Definition 27 (Step Functions)

A function $\varphi : \mathbb{R} \to \mathbb{R}$ is called a ***step function*** *if φ is simple with standard form*

$$\varphi = \sum a_n \chi_{E_n},$$

where each E_n is an interval or a singleton.

Recall the Definition . Given a bounded function $f : [a, b] \to \mathbb{R}$, we shall see that we can equivalently define

$$\overline{\int_a^b} f(x)\, dx = \inf \left\{ \int_{[a,b]} \varphi \, d\mu : f \leq \varphi, \, \varphi \text{ step function} \right\},$$

and

$$\underline{\int_a^b} f(x)\, dx = \sup \left\{ \int_{[a,b]} \varphi \, d\mu : \varphi \leq f, \, \varphi \text{ step function} \right\}.$$

We may then say that f is Riemann integrable if

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx.$$

We shall call the common value above as the **Riemann integral**, of which we shall denote by

$$\int_a^b f\,dx.$$

Definition 28 (Upper and Lower Envelopes of a Function)

Let $f : [a, b] \to \mathbb{R}$. We define

$$U_f(x) = U(x) = \lim_{\delta \downarrow 0} \sup_{|y-x| \leq \delta} f(y) = \max\{f(x), \limsup_{y \to x} f(y)\}$$

as the ***upper envelope of f****. We define*

$$L_f(x) = L(x) = \lim_{\delta \downarrow 0} \inf_{|y-x| \leq \delta} f(y) = \min\{f(x), \liminf_{y \to x} f(y)\}$$

as the ***lower envelope of f****.*

Proposition 43 (Characterization of Continuity with Upper and Lower Envelopes)

Let $f : [a, b] \to \mathbb{R}$. Then $U(x) = L(x)$ iff f is continuous at x.

Proof

$(\impliedby)$
f is continuous at x
$\iff f(x) = \lim_{y \to x} f(y)$
$\iff \limsup_{y \to x} f(y) = f(x) = \liminf_{y \to x} f(y)$
$\implies U(x) = f(x) = L(x)$.

$(\implies)$
$U(x) = L(x)$

$\iff f(x) \leq U(x) = L(x) \leq f(x)$

$\iff$

$$\limsup_{y \to x} f(y) \leq U(x) = f(x) = L(x) \leq \liminf_{y \to x} f(y) \leq \limsup_{y \to x} f(y)$$

$\iff \limsup_{y \to x} f(y) = \liminf_{y \to x} f(y)$

$\iff f(x) = \lim_{y \to x} f(y)$ □

Proposition 44 (Monotonic Sequence of Step Functions to the Upper and Lower Envelopes)

Let $f : [a,b] \to \mathbb{R}$. Then there exists step functions $\{\varphi_n\}_n$ such that $\varphi_n \nearrow L$, and step functions $\{\psi_n\}_n$ such that $\psi_n \searrow U$. Hence U and L are measurable.

Exercise 16.1.1

Prove Proposition 44. Hint: Take a partition of the domain, take refinements, ...

Theorem 45 (Characterization of the Upper and Lower Riemann Integrals of Bounded Functions)

Let $(X, \mathfrak{M}, \mu)$ be a Lebesgue measure space. Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then

$$\overline{\int_a^b} f \, dx = \int_{[a,b]} U \, d\mu,$$

and

$$\underline{\int_a^b} f \, dx = \int_{[a,b]} L \, d\mu.$$

Exercise 16.1.2

Prove Theorem 45.

Theorem 46 (Agreement of Riemann Integration and Lebesgue Integration for Bounded Functions)

Let $f : [a, b] \to \mathbb{R}$ be bounded. Then

1. *if f is Riemann integrable, then f is measurable and*

$$\int_a^b f\,dx = \int_{[a,b]} f\,d\mu.$$

2. *f is Riemann integrable iff*

$$\mu(\{x : f \text{ is not continuous at } x\}) = 0.$$

Proof

Will come back to this. □

16.2 *Modes of Convergences*

Notation

Consider a sequence of functions $\{f_n\}_n$.

- *(**Pointwise convergence**) We write $f_n \overset{ptw}{\to} f$ if $\forall x$ we have*

$$\lim_n f_n(x) = f(x).$$

- *(**Almost everywhere pointwise convergence**) We write $f_n \overset{a.e.}{\to} f$ if*

$$\lim_n f_n(x) = f(x)$$

  except for x's in a set of measure zero.

- *(Uniform convergence) We write $f_n \overset{unif}{\to} f$ if*

$$\forall x > 0 \, \exists N \in \mathbb{N} \, \forall x \, \forall n > N \, \left| f_n(x) - f(x) \right| < \varepsilon.$$

Definition 29 ($\mathcal{L}^1$-convergence)

*For $f_n, f \in \mathcal{L}^1$, we say that the f_n's **converge in $\mathcal{L}^1$** to f, of which we denote by*

$$f_n \overset{\mathcal{L}^1}{\to} f,$$

when

$$\int_X \left| f_n - f \right| d\mu \to 0.$$

Example 16.2.1

Consider the Lebesgue measurable space $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \mu)$.

1. Let $f_n = \frac{1}{n}\chi_{(0,n)}$ for each n, and $f \equiv 0$. We see that

   - $f_n \overset{\text{ptw}}{\to} 0$ (once $x > n$),
   - $f_n \overset{\text{a.e.}}{\to} 0$, and
   - $f_n \overset{\text{unif}}{\to} 0$.

   However, $\forall n$,

$$\int_X \left| f_n - 0 \right| d\mu = \frac{1}{n} \cdot n = 1 \to 1.$$

   Thus $f_n \overset{\mathcal{L}^1}{\not\to} 0$.

Summary for modes of convergences in example 1.

$$f_n \overset{\text{ptw}}{\to} 0 \qquad f_n \overset{\text{a.e.}}{\to} 0$$
$$f_n \overset{\text{unif}}{\to} 0 \qquad f_n \overset{\mathcal{L}^1}{\not\to} 0$$

2. Let $f_n = \chi_{[n,n+1)}$ for each n, and $f \equiv 0$. We see that

   - $f_n \overset{\text{ptw}}{\to} 0$ (same reason as before), and
   - $f_n \overset{\text{a.e.}}{\to} 0$.

   However, $f_n \overset{\text{unif}}{\not\to} 0$, since $\forall \varepsilon > 0 \, \forall N, \exists x$ and $\exists m > N$ such that $f_m(x) \neq 0$.

   Also, for each n,

$$\int \left| f_n - 0 \right| d\mu = 1 \to 1 \neq 0.$$

Summary for modes of convergences in example 2.

$$f_n \overset{\text{ptw}}{\to} 0 \qquad f_n \overset{\text{a.e.}}{\to} 0$$
$$f_n \overset{\text{unif}}{\not\to} 0 \qquad f_n \overset{\mathcal{L}^1}{\not\to} 0$$

Thus $f_n \overset{\mathcal{L}^1}{\not\to} 0$.

3. Let $f_n = n\chi_{[0,\frac{1}{n}]}$ for each n, and $f \equiv 0$.

   Then $f_n \overset{\text{ptw}}{\not\to} 0$ since for $x = 0$, $\forall \varepsilon > 0$, for any $N \in \mathbb{N}$, there always exists $n_0 > N$ such that $f_{n_0}(x) = n_0 \neq 0$.

   Now notice that the above is only the case for $x = 0$, and singletons have measure zero under the Lebesgue measure. Thus $f_n \overset{\text{a.e.}}{\to} 0$.

   For uniform convergence, by the reason stated for when $f_n \overset{\text{ptw}}{\not\to} 0$, we know that there is no 'uniform' $\varepsilon > 0$ that will give us what is required for this mode of convergence. Thus $f_n \overset{\text{unif}}{\not\to} 0$.

   For $\mathcal{L}^1$-convergence, since for $n \geq 1$,

$$\int_{[0,1]} f_n \, d\mu = n \cdot \frac{1}{n} = 1,$$

   we have that

$$\int_{[0,1]} \left| f_n - 0 \right| d\mu = 1 \to 1 \neq 0.$$

   Thus $f_n \overset{\mathcal{L}^1}{\not\to} 0$.

Summary for modes of convergences in example 3.

$$f_n \overset{\text{ptw}}{\not\to} 0 \qquad f_n \overset{\text{a.e.}}{\to} 0$$
$$f_n \overset{\text{unif}}{\not\to} 0 \qquad f_n \overset{\mathcal{L}^1}{\not\to} 0$$

4. Consider the following sequence of functions of which we have no nice way to properly express recursively so.

$$\begin{gathered} f_1 = \chi_{[0,1]}, \\ f_2 = \chi_{[0,\frac{1}{2}]}, \quad f_3 = \chi_{[\frac{1}{2},1]}, \\ f_4 = \chi_{[0,\frac{1}{4}]}, \quad f_5 = \chi_{[\frac{1}{4},\frac{1}{2}]}, \quad f_6 = \chi_{[\frac{1}{2},\frac{3}{4}]}, \quad f_7 = \chi_{[\frac{3}{4},1]}, \\ \vdots \end{gathered}$$

   (Pointwise convergence) We observe that on any $x \neq \frac{1}{2^m}$ for any $m \geq 1$, $f_n(x) \to 0$. However, on $x = \frac{1}{2^m}$, for any $m \geq 1$, $f_n(x) = 1$. Thus, it is clear that $f_n \overset{\text{ptw}}{\not\to} 0$.

   (Pointwise convergence almost everywhere) [1]

   (Uniform convergence) We know that there is no $\varepsilon > 0$ such that $f_n(x) = 0$ for every x by the reason stated in (Pointwise convergence).

Summary for modes of convergences in example 4.

$$f_n \overset{\text{ptw}}{\not\to} 0 \qquad f_n \overset{\text{a.e.}}{\not\to} 0$$
$$f_n \overset{\text{unif}}{\not\to} 0 \qquad f_n \overset{\mathcal{L}^1}{\to} 0$$

[1] Requires clarification.
Current belief is that this should hold, by the reason stated in (Pointwise convergence), and that $\mu(\mathbb{Q}) = 0$. However, lecture recorded $f_n \overset{\text{a.e.}}{\not\to} 0$.

($\mathcal{L}^1$-convergence) We see that indeed

$$\int_{[0,1]} f_n \, d\mu \to 0,$$

since the integral of the functions have the form of $\frac{1}{n}$, although we see that the occurrence for each $\frac{1}{n}$ occurs longer and longer.

Thus $f_n \xrightarrow{\mathcal{L}^1} 0$.

Remark 16.2.1

We see that uniform convergence has no logical relationship with $\mathcal{L}^1$-convergence.

We introduce a new mode of convergence.

Definition 30 (Convergence in Measure)

Let $(X, \mathfrak{M}, \mu)$ be a measure space, and $f_n, f : X \to \mathbb{R}_e$ be measurable functions. We say that $\{f_n\}$ ***converges in measure*** *to f, of which we denote by*

$$f_n \xrightarrow{\mu} f,$$

provided that $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$

$$\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) < \varepsilon,$$

or equivalently

$$\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \to 0.$$

Exercise 16.2.1

We look back at the last example. Show that the following are the case with respect to each of the examples.

1. $f_n \xrightarrow{\mu} 0$
2. $f_n \not\xrightarrow{\mu} 0$
3. $f_n \xrightarrow{\mu} 0$

4. $f_n \xrightarrow{\mu} 0$

17 Lecture 17 Oct 11th 2019

17.1 Modes of Convergences (Continued)

Definition 31 (Cauchy in Measure)

Let $(X, \mathfrak{M}, \mu)$ be a measure space, and $f_n, f : X \to \mathbb{R}_e$ be measurable functions. We say that $\{f_n\}$ is ***Cauchy in measure*** *if $\forall \varepsilon_1, \varepsilon_2 > 0$, $\exists N \in \mathbb{N}$ such that $\forall n, m > N$, we have*

$$\mu(\{x : \left| f_n(x) - f_m(x) \right| \geq \varepsilon_1\}) < \varepsilon_2.$$

Proposition 47 ($\mathcal{L}^1$-convergence implies Convergence in Measure)

Let $(X, \mathfrak{M}, \mu)$ be a measure space, and $f_n : X \to \mathbb{R}_e$ a sequence of measurable functions. If $f_n \xrightarrow{\mathcal{L}^1} f$ for some function f, then $f_n \to f$ in measure.

Proof

For any $\varepsilon > 0$ and any $n \geq 1$, let

$$E_{n,\varepsilon} := \{x : \left| f_n(x) - f(x) \right| \geq \varepsilon\}.$$

Then, by assumption, we have that

$$\varepsilon\mu(E_{n,\varepsilon}) \leq \int_{E_{n,\varepsilon}} \left|f_n - f\right| d\mu \leq \int_X \left|f_n - f\right| d\mu \to 0.$$

It follows that indeed

$$\mu(E_{n,\varepsilon}) \to 0 \text{ as } n \to \infty. \qquad \square$$

Theorem 48 (Various results related to convergence in measure)

Let $(X, \mathfrak{M}, \mu)$ be a measure space, $\{f_n : X \to \mathbb{R}_e\}_n$ a sequence of measurable functions, and f, g be measurable functions.

1. $f_n \xrightarrow{\mu} f \wedge f_n \xrightarrow{\mu} g \implies f = g$ *a.e.*
2. $\{f_n\}_n$ *is Cauchy in measure* $\implies \exists f : X \to \mathbb{R}_e$ *measurable such that* $f_n \xrightarrow{\mu} f$.
3. $f_n \xrightarrow{\mu} f$ *in measure* $\implies \exists \{f_{n_k}\}_k \subseteq \{f_n\}_n$ *such that* $f_{n_k} \xrightarrow{a.e.} f$.

Proof

1. Suppose $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\mu} g$. Let $\varepsilon > 0$. Note that

   $$\left|f(x) - g(x)\right| \leq \left|f(x) - f_n(x)\right| + \left|f_n(x) - g(x)\right|.$$

   Thus

   $$\begin{aligned} &\{x : \left|f(x) - g(x)\right| \geq \varepsilon\} \\ &\subseteq \left\{x : \left|f(x) - f_n(x)\right| \geq \frac{\varepsilon}{2}\right\} \cup \left\{x : \left|f_n(x) - g(x)\right| \geq \frac{\varepsilon}{2}\right\}. \end{aligned}$$

   By monotonicity,

   $$\begin{aligned} &\mu(\{x : \left|f(x) - g(x)\right| \geq \varepsilon\}) \\ &\leq \mu\left(\left\{x : \left|f(x) - f_n(x)\right| \geq \frac{\varepsilon}{2}\right\}\right) + \mu\left(\left\{x : \left|f_n(x) - g(x)\right| \geq \frac{\varepsilon}{2}\right\}\right). \end{aligned}$$

   By our assumption, the terms on the RHS converges to 0 as $n \to$

∞. It follows that

$$\mu(\{x : \left|f(x) - g(x)\right| \geq \varepsilon\}) = 0$$

for arbitrary $\varepsilon > 0$.

Now notice that

$$\{x : f(x) \neq g(x)\} \subseteq \bigcup_n \left\{x : \left|f(x) - g(x)\right| > \frac{1}{n}\right\}.$$

It follows that

$$\mu(\{x : f(x) \neq g(x)\}) \leq \sum_n \mu\left(\left\{x : \left|f(x) - g(x)\right| > \frac{1}{n}\right\}\right) = 0.$$

Therefore $\mu(\{x : f(x) \neq g(x)\}) = 0$. Thus $f = g$ a.e.

2. **Something's wrong with the prove given in class. Will confirm.**

□

Homework (Homework 14)

Let $(X, \mathfrak{M}, \mu)$ be a measure space, $f_n, f : X \to \mathbb{R}_e$ for $n \geq 1$, with $f_n \to f$ a.e. Suppose $\mu(X) < \infty$. Furthermore, suppose that there exists an M such that $\forall n \geq 1$,

$$\mu(\{x : \left|f_n(x)\right| \geq M\}) = 0.$$

[1] *Prove that*

$$\int_X f \, d\mu = \lim_n \int_X f_n \, d\mu.$$

[1] We may say that each f_n is bounded by M a.e.

Homework (Homework 15)

Let $(X, \mathfrak{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let us say that $f \sim g$ when $f = g$ a.e. Let

$$[f] \coloneqq \{g : g \sim f\}.$$

Let

$$Y \coloneqq \{[f] : f : X \to \mathbb{R} \text{ is measurable }\}.$$

Prove that

1. *The function*
$$\rho([f],[g]) = \int_X \frac{|f-g|}{1+|f-g|}\,d\mu$$
is a metric on Y.

2. $\rho([f_n],[f]) \to 0$ *if and only if* $f_n \to f$ *in measure.*

Homework (Homework 16 (Generalized Fatou's Lemma))

Let $(X, \mathfrak{M}, \mu)$ be a measure space, $f_n, f : X \to \mathbb{R}_e$ for $n \geq 1$, with $f_n \geq 0$. If $f_n \to f$ in measure, then

$$\int_X f\,d\mu \leq \liminf_n \int_X f_n\,d\mu.$$

18 Lecture 18 Oct 23rd 2019

18.1 Modes of Convergences (Continued 2)

Theorem 49 (Egoroff's Theorem)

Let $(X, \mathfrak{M}, \mu)$ be a measure space, and $\mu(X) < \infty$. Let $f_n, f : X \to \mathbb{R}$ be measurable functions. Suppose $f_n \overset{a.e.}{\to} f$. Then given $\varepsilon > 0$, $\exists E \in \mathfrak{M}$ such that $\mu(E) < \varepsilon$ with $f_n \overset{unif}{\to} f$ on E^C.

Proof

Suppose $f_n \overset{\text{a.e.}}{\to} f$. Let

$$N := \left\{ x : f_n(x) \overset{\text{ptw}}{\not\to} f(x) \right\}$$

be the no-good set. Then $X = N \cup X_1$, where

$$X_1 := \left\{ x : f_n(x) \overset{\text{ptw}}{\to} f(x) \right\}.$$

Let

$$E_{n,k} := \bigcup_{m=n}^{\infty} \left\{ x \in X_1 : \left| f_m(x) - f(x) \right| \geq \frac{1}{k} \right\}.$$

Notice that $E_{n,k} \supseteq E_{n+1,k}$ and

$$\bigcap_{n=1}^{\infty} E_{n,k} = \emptyset.$$

By continuity from above,

$$\lim_n \mu(E_{n,k}) = \mu(\emptyset) = 0.$$

Thus we may pick a subsequence n_l such that

$$\mu(E_{n_l}, l) < \frac{\varepsilon}{2^l}.$$

Let

$$E = \bigcup_l E_{n_l,l}.$$

Then by monotonicity and subadditivity of measures,

$$\mu(E) \leq \sum_l \mu(E_{n_l,l}) < \varepsilon.$$

Let $\varepsilon > 0$. Choose $L > 0$ such that $\frac{1}{L} < \varepsilon$. Now for any $x \in E^C$, we have

$$x \in \bigcap_l E^C_{n_l,l}.$$

Note that

$$E^C_{n_l,l} = \bigcap_{m=n_l}^{\infty} \left\{ x : \left| f_m(x) - f(x) \right| \leq \frac{1}{l} \right\}.$$

Therefore, for any $l > L$, for any $m \geq n_l$, we have

$$\left| f_m(x) - f(x) \right| \leq \frac{1}{l} < \frac{1}{L} < \varepsilon.$$

Therefore, indeed $f_n \overset{\text{unif}}{\to} f$ on E^C. □

18.2 Product Measures

Note 18.2.1 (Motivation for looking at Product Measures of σ-finite Measures)

Given measure spaces $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$,

1. *we want a measure space of the form $(X \times Y, \mathfrak{M} \otimes \mathfrak{N}, \lambda)$ where*

$$\lambda(A \times B) = \mu(A) \cdot \nu(B),$$

and

2. *to have the measure play nice with integration, in that given a* $(\mathfrak{M} \otimes \mathfrak{N})$*-measurable function* $f : X \times Y \to [0, \infty]$*, we want*

$$\begin{aligned} \int_{X \times Y} f \, d\lambda &= \int_Y \left[\int_X f \, d\mu \right] d\nu \\ &= \int_X \left[\int_Y f \, d\nu \right] d\mu. \end{aligned}$$

It turns out that we can always have (1) but not (2). We will not go into the details of showing for (1) in full detail. However, we can have both when μ *and* ν *are* σ*-finite.*

Note 18.2.2 (A sketch of how we can always construct λ)

Given any $E \subseteq X \times Y$*, we define the function*

$$\lambda^*(E) := \inf \left\{ \sum_n \mu(A_n) \cdot \nu(B_n) : E \subseteq \bigcup_n A_n \times B_n, \, A_n \in \mathfrak{M}, \, B_n \in \mathfrak{N} \right\}.$$

Exercise 18.2.1

Prove that λ^* *is an outer measure.*

By Carathéodory's Theorem, the set

$$\mathcal{L} := \{ A \times B \subseteq X \times Y : A \times B \text{ is } \lambda^*\text{-measurable} \}$$

is a σ*-algebra, and we can thus define* λ *as a complete measure on* $\mathcal{L}$*. It then remains to show that* $\mathfrak{M} \otimes \mathfrak{N} \subseteq \mathcal{L}$*.*

Example 18.2.1 (How (2) fails when one of the measures is not σ-finite)

Consider $X = Y = [0, 1]$, $\mathfrak{M} = \mathfrak{N} = \mathfrak{B}(X) = \mathfrak{B}(Y)$, and $\mathfrak{M}$ is the Lebesgue measure while ν is the counting measure. We know that ν is not σ-finite.

Consider
$$D = \{(t,t) : t \in [0,1]\} \subseteq [0,1] \times [0,1],$$
which is Borel and hence measurable. Then we can define
$$\lambda(D) = \lambda^*(D) := \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) \cdot \nu(B_n) : D \subseteq \bigcup_n A_n \times B_n \right\}.$$
Note that the components of each element in D iterates through every real number in $[0,1]$. Furthermore, $[0,1] \subseteq \bigcup_n B_n$, and so we have $\nu(B_n) = \infty$. We can then show that $\lambda(D) = \infty$. Then
$$\int_{[0,1]\times[0,1]} \chi_D \, d\lambda = \lambda(D) = \infty.$$
However, we see that
$$\int_{[0,1]} \int_{[0,1]} \chi_D \, d\mu \, d\nu = \int_{[0,1]} 0 \, d\nu = 0,$$
since when we fix one value of y, $\chi_D(x,y) = 1$ only happens at one point that is $x = y$. Furthermore, by a similar reasoning, when we fix x, $\chi_D(x,y) = 1$ iff $x = y$ and so
$$\int_{[0,1]} \chi_D \, d\nu = \nu(\{x\}) = 1,$$
when then we see
$$\int_{[0,1]} \int_{[0,1]} \chi_D \, d\nu \, d\mu = \int_{[0,1]} 1 \, d\nu = 1,$$

Proposition 50 (Component-wise Measurability of Functions and Sets)

Let $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ be measurable sets.

1. *If $E \in \mathfrak{M} \otimes \mathfrak{N}$, then*
$$E_x = \{y \in Y : (x,y) \in E\} \in \mathfrak{N},$$
$$E^y = \{x \in X : (x,y) \in E\} \in \mathfrak{M}.$$

2. *If $f : X \times Y \to \mathbb{R}_e$ is $(\mathfrak{M} \otimes \mathfrak{N})$-measurable, then*

$$f_x(y) = f(x,y) : Y \to \mathbb{R}_e \text{ is } \mathfrak{N}\text{-measurable}$$
$$f^y(x) = f(x,y) : X \to \mathbb{R}_e \text{ is } \mathfrak{M}\text{-measurable}$$

Proof

1. This was Homework 9.

2. Fix a. Let $E = (a, \infty]$. Since f is $\mathfrak{M} \otimes \mathfrak{N}$-measurable, we have that $f^{-1}(E) \in \mathfrak{M} \otimes \mathfrak{N}$. It follows from part (1) that if we fix x,

$$f_x^{-1}(E) = \{y : f(x,y) \in E\} = E_x \in \mathfrak{N}.$$

The result holds similarly for $f^y(E) \in \mathfrak{M}$. $\square$

Definition 32 (Monotone Classes)

*Let X be a non-empty set. $\xi \subseteq \mathcal{P}(X)$ is called a **monotone class** if*

1. *$E_j \in \xi$ with $E_1 \subseteq E_2 \subseteq \ldots \implies \bigcup_n E_n \in \xi$, and*

2. *$F_n \in \xi$ with $F_1 \supseteq F_2 \supseteq \ldots \implies \bigcap_n F_n \in \xi$.*

Note 18.2.3

1. *Notice that every σ-algebra is a monotone class.*

2. *$\mathcal{P}(X)$ is a monotone class.*

3. *If $\xi_\alpha \subseteq \mathcal{P}(X)$ are monotone classes, then*

$$\bigcap_\alpha \xi_\alpha = \{E \subseteq X : E \in \xi_\alpha, \forall \alpha\}$$

is also a monotone class.

4. *Given any collection of subsets of X, there exists a smallest monotone class that contains them, and we call this the monotone class generated by those subsets.*

Theorem 51 (Monotone Class Theorem)

Let X a non-empty set, and $A \subseteq \mathcal{P}(X)$ an algebra of sets. Then the σ-algebra generated by A is equal to the monotone class generated by A.

We require the following lemma.

Lemma 52 (Lemma for Monotone Class Theorem)

Suppose $\mathcal{A}$ is an algebra of sets on X. Then $\mathcal{A}$ is a σ-algebra iff $\forall \{E_j\} \subseteq \mathcal{A}$ with $E_1 \subseteq E_2 \subseteq \ldots$, then $\bigcup_j E_j \in \mathcal{A}$.

Proof

($\implies$) Follows simply by definition of a σ-algebra.

($\impliedby$) Let $\{A_i\}_i \subseteq \mathcal{A}$. Consider

$$E_1 = A_1, \, E_2 = A_1 \cup A_2, \, \ldots .$$

Then by assumption,

$$\bigcup_i A_i = \bigcup_i E_i \in \mathcal{A}$$

since $E_1 \subseteq E_2 \subseteq \ldots$. Thus $\mathcal{A}$ is indeed a σ-algebra. $\square$

19 Lecture 19 Oct 25th 2019

19.1 Product Measures (Continued)

Proof (Proof for Monotone Class Theorem)

Let ζ be the monotone class generated by $\mathcal{A}$ and $\mathfrak{M}$ be the σ-algebra generated by $\mathcal{A}$. We want to show that $\zeta = \mathfrak{M}$.

Note that since every σ-algebra is a monotone class, we must have $\zeta \subseteq \mathfrak{M}$. Thus it remains to show that $\mathfrak{M} \subseteq \zeta$. To that end, it suffices for us to show that ζ is a σ-algebra, since $\mathfrak{M}$ is the "smallest" σ-algebra generated by $\mathcal{A}$.

[1] Let $E \in \zeta$. Set

[1] The proof gets really slippery from hereon.

$$\zeta_E \coloneqq \{F \in \zeta : F \setminus E,\, E \setminus F,\, E \cap F \in \zeta\}.$$

We want to show that $\zeta_E = \zeta$.

It is clear that $E \in \zeta_E$. Furthermore, $\emptyset \in \zeta_E$ since $E = E \setminus \emptyset \in \zeta$. Note that $F \in \zeta_E \iff E \in \zeta_F$.

Claim: ζ_E is a monotone class Suppose $F_j \in \zeta_E$ with $F_1 \subseteq F_2 \subseteq \ldots$. Let $F = \bigcup_j F_j$. For each j, we know that

$$F_j \setminus E,\, E \setminus F_j,\, E \cap F_j \in \zeta.$$

Thus since ζ is a monotone class, and

$$F_1 \setminus E \subseteq F_2 \setminus E \subseteq \ldots,$$

we have

$$F \setminus E = \bigcup_j (F_j \setminus E) \in \zeta.$$

Since

$$E \setminus F_1 \supseteq E \setminus F_2 \supseteq \ldots,$$

we also have

$$E \setminus F = E \setminus \bigcup_j F_j = E \cap \left(\bigcap_j F_j^C \right) = \bigcap_j E \setminus F_j \in \zeta.$$

Finally,

$$E \cap F_1 \subseteq E \cap F_2 \subseteq \ldots,$$

so

$$E \cap \bigcup_j F_j \in \zeta.$$

Therefore, $F \in \zeta_E$. Similarly, given $F_1 \supseteq F_2 \supseteq \ldots$, we can show that $\bigcap_j F_j \in \zeta_E$. This proves the claim. $\dashv$

$\zeta_A = \zeta$ for any $A \in \mathcal{A}$ It is clear that $\zeta_A \subseteq \zeta$ simply by definition.

It remains to show that $\zeta \subseteq \zeta_A$. To that end, we simply need to look at elements from the generator $\mathcal{A}$. Let $E \in \mathcal{A} \subseteq \zeta$. Since $\mathcal{A}$ is an algebra of sets, and $A \in \mathcal{A}$, we have that $E \setminus A$, $A \setminus E$, and $A \cap E \in \mathcal{A} \subseteq \zeta$. Thus $E \in \zeta_A$, and furthermore, $\mathcal{A} \subseteq \zeta_A$. This proves the claim. $\dashv$

$\zeta_E = \zeta$ for any $E \in \zeta$ We know that for any $A \in \mathcal{A}$, $E \in \zeta_A \iff A \in \zeta_E$. Thus $\mathcal{A} \subseteq \zeta_E$. By the last claim, we know that $\zeta_E = \zeta$ for any $E \in \zeta$. $\dashv$

We now know that $\forall E, F \in \zeta$,

$$F \in \zeta_E \Rightarrow E \setminus F,\ F \setminus E,\ E \cap F \in \zeta.$$

Thus, for any $E \in \zeta$, letting $F = X$, we have that

$$F \setminus E = E^C,\ E \setminus F = \emptyset,\ E \cap F = E \in \zeta.$$

In particular, $\emptyset$, $E^C \in \zeta$.

Now $\forall E, , F \in \zeta$, we have $E^C, F^C \in \zeta$, and so $E^C \cap F^C \in \zeta$. Thus

$$E \cup F = (E^C \cap F^C)^C \in \zeta.$$

By induction, ζ is closed under finite unions, and by De Morgan's Laws, closed under finite intersections. Thus ζ is an algebra of sets and a monotone class.

By Lemma 52, we have that ζ is a σ-algebra. It follows that $\mathfrak{M} \subseteq \zeta$, and our proof is done. □

Theorem 53 (Existence of Product Measures for σ-finite Measure Spaces)

Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be σ-finite measure spaces. Let $E \in \mathfrak{M} \otimes \mathfrak{N}$. Then

1. *$x \mapsto \nu(E_x)$ is $\mathfrak{M}$-measurable and $y \mapsto \mu(E^y)$ is $\mathfrak{N}$-measurable;*
2. *we have*
$$\int_X \nu(E_x)\, d\mu = \int_Y \mu(E^y)\, d\nu;$$
3. *if we set $(\mu \times \nu)(E) = \int_X \nu(E_x)\, d\mu$, then $\mu \times \nu$ is a measure on $\mathfrak{M} \otimes \mathfrak{N}$ and*
$$(\mu \times \nu)(A \times B) = \mu(A) \cdot \nu(B).$$

Note 19.1.1
It is often useful to think of $A \times B$ as a rectangle.

Proof

Case: $\mu(X), \nu(Y) < \infty$ Let

$$\zeta := \{E \in \mathfrak{M} \otimes \mathfrak{N} : \text{ (1) and (2) are true } \}.$$

Let $\mathcal{A}$ be the algebra generated by $\{A \times B : A \in \mathfrak{M}, B \in \mathfrak{N}\}$.

[2] $F \in \mathcal{A} \implies F = \bigcup_{i=1}^{n} (A_i \times B_i)$, where $A_i \in \mathfrak{M}$, $B_i \in \mathfrak{N}$ Note that for any $F \in \mathcal{A}$, we may write $F = A \times B$ for some $A \in \mathfrak{M}$ and $B \in \mathfrak{N}$. As stated, note that $\mathcal{A}$ is an algebra.

[2] We want to show that we may write each element of $\mathcal{A}$ as a disjoint union of rectangles, each of which has components in $\mathfrak{M}$ and $\mathfrak{N}$.

If $F = (A \times B)^C$, then we may write

$$(A \times B)^C = (A^C \times Y) \mathbin{\dot\cup} (A \times B^C)$$

(see Figure 19.1).

If $F = (A_1 \times B_1) \cup (A_2 \times B_2)$, then we may partition F such that

$$\begin{aligned} &(A_1 \times B_1) \cup (A_2 \times B_2) \\ &= (A_1 \cap A_2) \times (B_1 \cap B_2) \mathbin{\dot\cup} (A_1 \setminus A_2) \times B_1 \mathbin{\dot\cup} (A_2 \setminus A_1) \times B_2 \end{aligned}$$

(see Figure 19.2).

By extending on the above inductively, we can prove that we may write

$$F = \dot{\bigcup_{i=1}^{n}} (A_i \times B_i)$$

for any $F \in \mathcal{A}$, with $A_i \in \mathfrak{M}$ and $B \in \mathfrak{N}$. ⊣

Require clarification before proceeding, cause I have no idea why we even did anything.

□

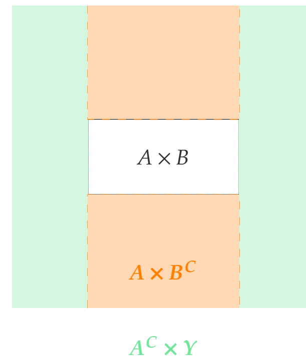

Figure 19.1: Idea of partitioning $(A \times B)^C$.

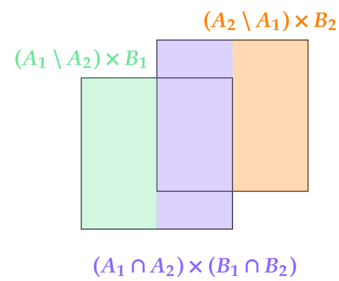

Figure 19.2: Idea of partitioning $(A_1 \times B_1) \times (A_2 \times B_2)$.

20 Lecture 20 Oct 28th 2019

20.1 Product Measures (Continued 2)

Theorem 54 (Fubini-Tonelli Theorem)

Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be σ-finite measure spaces.

1. *(Tonelli) If $f : X \times Y \to \mathbb{R}_e$ with $f \geq 0$ is a $\mathfrak{M} \otimes \mathfrak{N}$-measurable function, then*

$$g(x) = \int_Y f_x \, d\nu \text{ is } \mathfrak{M}\text{-measurable},$$

and

$$h(x) = \int_X f^y \, d\mu \text{ is } \mathfrak{N}\text{-measurable}.$$

Furthermore,

$$\int_X \int_Y f_x \, d\nu \, d\mu = \int_Y \int_X f^y \, d\nu = \int_{X \times Y} f \, d(\mu \times \nu).$$

2. *(Fubini) If $f : X \times Y \to \mathbb{R}_e \in \mathcal{L}^1(X \times Y)$ is $(\mathfrak{M} \otimes \mathfrak{N})$-measurable, then*

$$f_x \in \mathcal{L}^1(Y) \text{ for a.e. } x \quad \text{and} \quad f^y \in \mathcal{L}^1(X) \text{ for a.e. } y,$$

and

$$\int_X \int_Y f_x \, d\nu \, d\mu = \int_Y \int_X f^y \, d\mu \, d\nu = \int_{X \times Y} f \, d(\mu \times \nu).$$

Let us first consider several relevant examples.

Example 20.1.1

Consider $X = Y = \mathbb{R}$ and $\mu = \nu$ is the Lebesgue measure. Consider the function

$$f = \chi_{\{x_0\} \times \mathbb{R}} + \chi_{\mathbb{R} \times \{y_0\}},$$

for some $x_0, y_0 \in \mathbb{R}$. We see that $f = 0$ a.e., and so $f \in \mathcal{L}^1(\mathbb{R} \times \mathbb{R})$. However,

$$f_{x_0}(y) = 1 \;\; \forall y \implies f_{x_0} \notin \mathcal{L}^1(\mathbb{R}).$$

Similarly,

$$f^{y_0}(x) = 1 \;\; \forall x \implies f^{y_0} \notin \mathcal{L}^1(\mathbb{R}).$$

Tonelli's statement says that such points x_0 and y_0, in their respective spaces, can only happen on a set of measure zero.

Example 20.1.2 (Failure of Fubini's Theorem when $f \notin \mathcal{L}^1(X \times Y)$)

Consider the function

$$f(x, y) = \begin{cases} 1 & x \leq y \leq x + 1 \\ -1 & x - 1 \leq y < x \\ 0 & \text{otherwise} \end{cases}.$$

Note that the graph of f is illustrated in Figure 20.1.

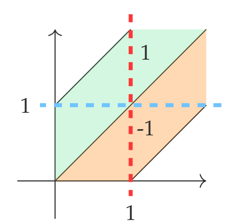

Figure 20.1: Values of $f(x, y)$ in Example 20.1.2

Notice that when $x \geq 1$ (to the right of the red dashed vertical line), we have

$$\int_{\mathbb{R}} f_x \, dy = 0.$$

When $x < 0$, we have

$$\int_{\mathbb{R}} f_x \, dy = 0.$$

When $x = 0$, we have

$$\int_{\mathbb{R}} f_x \, dy = 1.$$

Finally, when $x \in (0, 1)$, we have

$$0 < \int_{\mathbb{R}} f_x \, dy < 1.$$

This holds rather similarly for y for f^y (using the blue dashed horizontal line for reference), except for the fact that

$$-1 < \int_{\mathbb{R}} f^y \, dx < 0$$

for when $y \in (0,1)$, and

$$\int_{\mathbb{R}} f^0 \, dx = -1$$

for when $y = 0$.

In particular, one may see that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f_x \, dy \, dx = \frac{1}{2}.$$

However,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f^y \, dx \, dy = -\frac{1}{2}.$$

Proof (Tonelli)

Case 1: Characteristic Functions Suppose $f = \chi_E$ where $E \in \mathfrak{M} \otimes \mathfrak{N}$. Fixing x, we have

$$f_x(y) = \chi_E(x,y) = \begin{cases} 1 & (x,y) \in E \\ 0 & (x,y) \notin E \end{cases} = \chi_{E_x}(y).$$

Thus $f_x = \chi_{E_x}$ and similarly $f^y = \chi_{E^y}$. Then by Theorem 53, we have that

$$\int_X \int_Y \chi_{E_x} \, d\nu \, d\mu = \int_Y \int_X \chi_{E^y} \, d\nu = \int_{X \times Y} \chi_E \, d(\mu \times \nu).$$

Case 2: Simple Functions This is simply extending on Case 1 by linearity of integration, since simple functions are expressible as finite sums.

Case 3: $(\mathfrak{M} \otimes \mathfrak{N})$-measurable Functions Since $f \geq 0$, we may construct $\varphi_n \nearrow f$ by way of Theorem 25. The result follows from the Monotone Convergence Theorem (MCT). □

21 Lecture 21 Oct 30th 2019

21.1 Product Measures (Continued 3)

Note 21.1.1 (Measure zero with respect to a certain measure)

*Given a measure μ, we shall say that a set A is a **μ-null** set if $\mu(A) = 0$, i.e. A has measure zero with respect to μ.*

Proof (Fubini)

Notice that since $\left|f\right| = f^+ + f^-$ while $f = f^+ - f^-$, we noted that $f \in \mathcal{L}^1(X \times Y) \iff \left|f\right| \in \mathcal{L}^1(X \times Y)$. In particular, $f^+, f^- \geq 0$ by construction. We notice that we can use Tonelli's theorem if we fulfill the rest of its conditions.

Recall that

$$\int_{X\times Y} f\, d(\mu \times \nu) = \int_{X\times Y} f^+\, d(\mu \times \nu) - \int_{X\times Y} f^-\, d(\mu \times \nu).$$

Let

$$g^+(x) = \int_Y f^+(x, y)\, d\nu \quad \text{and} \quad g^-(x) = \int_Y f^-(x, y)\, d\nu.$$

Then by Tonelli's theorem,

$$\int_{X\times Y} f^+\, d(\mu \times \nu) = \int_X g^+(x)\, d\mu$$

and

$$\int_{X\times Y} f^-\, d(\mu \times \nu) = \int_X g^-(x)\, d\mu.$$

Consider
$$N^+ \coloneqq \{x : g^+(x) = \infty\}.$$

Then since g^+ is $\mathfrak{M}$-measurable, we have that $\mu(N^+) = 0$. Similarly, the set $N^- \coloneqq \{x : g^-(x) = \infty\}$ is a μ-null set. It follows that $N = N^+ \cup N^-$ is also μ-null.

For $x \notin N$, let $g(x) = g^+(x) - g^-(x)$, which then
$$g(x) = \int_Y f(x,y)\, d\nu.$$

By Tonelli's theorem, we know that
$$\int_{X \setminus N} g^+ \, d\mu = \int_{X \times Y} f^+ \, d(\mu \times \nu) = \int_{(X \setminus N) \times Y} f^+ \, d(\mu \times \nu),$$

and
$$\int_{X \setminus N} g^- \, d\mu = \int_{X \times Y} f^- \, d(\mu \times \nu) = \int_{(X \setminus N) \times Y} f^- \, d(\mu \times \nu).$$

Thus
$$\begin{aligned}
\int_{X \times Y} f \, d(\mu \times \nu) &= \int_{(X \setminus N) \times Y} f^+ \, d(\mu \times \nu) - \int_{(X \setminus N) \times Y} f^- \, d(\mu \times \nu) \\
&= \int_{X \setminus N} g^+ \, d\mu - \int_{X \setminus N} g^- \, d\mu \\
&= \int_{X \setminus N} (g^+ - g^-) \, d\mu \\
&= \int_{X \setminus N} \left[\int_Y (f^+(x,y) - f^-(x,y)) \, d\nu \right] d\mu \\
&= \int_{X \setminus N} \left[\int_Y f(x,y) \, d\nu \right] d\mu \\
&= \int_X \left[\int_Y f(x,y) \, d\nu \right] d\mu.
\end{aligned}$$

The other iteration follows if we instead constructed
$$h^+(x) = \int_X f^+(x,y)\, d\mu \quad \text{and} \quad h^-(x) = \int_X f^-(x,y)\, d\mu$$

in place of g^+ and g^-, and follow through the proof similarly. □

Remark 21.1.1

Given σ-finite measure spaces $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$, we now know that we can get the measure $\mu \times \nu$ on $\mathfrak{M} \otimes \mathfrak{N}$.

We can, in fact, complete this measure, by making use of Theorem 6. In particular, we let

$$\widetilde{\mathfrak{M} \otimes \mathfrak{N}} := \{E \cup F : E \in \mathfrak{M} \otimes \mathfrak{N},\ F \subseteq B,\ B \in \mathfrak{M} \otimes \mathfrak{N},\ \text{such that } \mu \times \nu(B) = 0\}.$$

and let

$$\widetilde{\mu \times \nu}(E \cup F) = \mu \times \nu(E)$$

for all F's as described in $\widetilde{\mathfrak{M} \otimes \mathfrak{N}}$.

Example 21.1.1

It is important to note that even if μ and ν are both complete, $\mu \times \nu$ is almost never complete.

Consider $A \in \mathfrak{M}$ with $\mu(A) = 0$, and $\mathfrak{N} \neq \mathcal{P}(Y)$, where $\mu = \nu$ is the Lebesgue measure. If $E \in \mathcal{P}(Y) \setminus \mathfrak{N}$, then $A \times E \notin \mathfrak{M} \otimes \mathfrak{N}$. However, $A \times E \subseteq A \times Y$ and $\mu \times \nu(A \times Y) = 0$.

The following is a corollary for Theorem 54, stated as a theorem.

Theorem 55 (Fubini-Tonelli Theorem for Complete Measures)

Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be complete, σ-finite measure spaces. Let $(X \times Y, \mathfrak{O}, \lambda)$ be the completion of $(X \times Y, \mathfrak{M} \otimes \mathfrak{N}, \mu \times \nu)$. Let $f : X \times Y \to \mathbb{R}_e$ be $(\mathfrak{M} \otimes \mathfrak{N})$-measurable. Then f_x is $\mathfrak{N}$-measurable for a.e. x and f^y is $\mathfrak{M}$-measurable for a.e. y. Furthermore,

$$\int_X \int_Y f_x \, d\nu \, d\mu = \int_Y \int_X f^y \, d\mu \, d\nu = \int_{X \times Y} f \, d(\mu \times \nu).$$

Example 21.1.2

Need clarification

Consider $X = Y = \mathbb{R}$, and μ the Lebesgue measure. Let $\mathfrak{O} = \widetilde{\mathfrak{B}(\mathbb{R})}$.

Consider the measure spaces

$$(\mathbb{R}^2, \mathfrak{B}(\mathbb{R}) \otimes \mathfrak{B}(\mathbb{R}), \mu \times \mu) \quad \text{and} \quad (\mathbb{R}^2, \mathfrak{O} \otimes \mathfrak{O}, \mu \times \mu).$$

Homework (Homework 17)

Show that the conclusion of Egoroff's Theorem can fail to hold when $(X, \mathfrak{M}, \mu)$ is σ-finite.

Homework (Homework 18)

Let $f, f_n : \mathbb{R} \to \mathbb{R}$, each f_n continuous and

$$f(x) = \lim_n f_n(x) \quad \forall x.$$

Prove that $\forall \varepsilon > 0$, $\exists E \subseteq \mathbb{R}$ measurable, with $\mu(E) < \varepsilon$ such that f is continuous on E^C.

Homework (Homework 19)

Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be σ-finite measure spaces. Let $E \in \mathfrak{M} \otimes \mathfrak{N}$ with $\mu \times \nu(E) = 1$. Prove that for $t > 0$

$$\mu(\{x : \nu(E_x) \geq t\}) \leq \frac{1}{t}.$$

Homework (Homework 20)

Let $(X, \mathfrak{M}, \mu)$ be a measure space, and $f_n, g_n, f, g : X \to \mathbb{R}$ be measurable functions. If $f_n \to f$ in measure, $g_n \to g$ in measure, prove that $f_n + g_n \to f + g$ in measure.

Homework (Homework 21)

Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$*, where* μ *is the counting measure.*

1. *Let* $f : \mathbb{N} \to [0, \infty)$*, with* $f(n) = a_n$*. Prove that*

$$\int_{\mathbb{N}} f \, d\mu = \sum_{n=1}^{\infty} a_n.$$

2. *Let* $f : \mathbb{N} \to \mathbb{R}$*, with* $f(n) = a_n$*. Prove that* $f \in \mathcal{L}^1(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu) \iff \sum_{n=1}^{\infty} a_n$ *converges absolutely and that in this case*

$$\int_{\mathbb{N}} f \, d\mu = \sum_{n=1}^{\infty} a_n.$$

Homework (Homework 22)

Let $g : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ *with*

$$g(n, m) = a_{n,m}.$$

For each finite subset $F \subseteq \mathbb{N} \times \mathbb{N}$*, set*

$$S_F = \sum_{(n,m) \in F} a_{n,m}.$$

We say that $\lim_F S_F$ *exists and is equal to* L *provided that* $\forall \varepsilon > 0$*,* $\exists F_0$ *a finite set such that* $\forall F \supseteq F_0$ *where* F *is finite, we have* $|L - S_F| < \varepsilon$*.*

Prove or disprove: $g \in \mathcal{L}^1(\mathbb{N} \times \mathbb{N}, \mathcal{P}(\mathbb{N}) \otimes \mathcal{P}(\mathbb{N}), \mu \times \mu) \iff \lim_F S_F$ *exists. And, in this case*

$$\int_{\mathbb{N} \times \mathbb{N}} g \, d(\mu \times \mu) = \lim_F S_F.$$

(You may use $\mathcal{P}(\mathbb{N}) \otimes \mathcal{P}(\mathbb{N}) = \mathcal{P}(\mathbb{N} \times \mathbb{N})$*.)*

21.2 Signed Measures

Consider a function $f < 0$. We know that we can seek the integral of f with respect to a given measure μ, i.e.

$$\int_X f \, d\mu.$$

Considering the way we define such an integral, we may very well have $\int_X f \, d\mu < 0$. On the other hand, we know that we can define a function

$$\nu(E) = \int_E f \, d\mu.$$

Had f be non-negative, ν would be a measure. We want to allow ν, where $f < 0$, to also be considered a 'measure'. This leads us to consider the following notion.

Definition 33 (Signed Measure)

Let $(X, \mathfrak{M})$ be a measurable space. We say that $\nu : X \to \mathbb{R}_e$ is a ***signed measure*** *if*

1. $\nu(\emptyset) = 0$;
2. *ν can take on all values in $\mathbb{R}_e$, but* ***not*** *both ∞ and $-\infty$; and*
3. $E = \dot{\bigcup}_{n=1}^{\infty} E_n \implies \nu(E) = \sum_{n=1}^{\infty} \nu(E_n)$, *where $\sum \nu(E_n)$ converges absolutely when $\nu(E) \neq \pm\infty$, and properly diverges to ∞ if $\nu(E) = \infty$ (i.e. $\forall c > 0, \exists N \in \mathbb{N}, \forall n \geq N, \sum_{j=1}^{n} \nu(E_j) > c$).*

Remark 21.2.1

The second condition where ν cannot simultaneously have $-\infty$ and ∞ in its domain is considered so that we do not have to deal with the controversial case of $\infty - \infty$, while still allowing for ∞ to be a possible value in our measurement.

Example 21.2.1

1. Consider $f \in \mathcal{L}^1(X, \mathfrak{M}, \mu)$ where μ is a (regular) measure. Note

$f = f^+ - f^-$. Define

$$\nu(E) = \int_E f \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu = \mu^+(E) - \mu^-(E),$$

where μ^+ and μ^- are ordinary measures, while ν is a signed measure.

2. Let μ_1, μ_2 be 2 measures on the measurable space $(X, \mathfrak{M})$. Wlog, suppose $\mu_1(X) < \infty$. Then

$$\nu(E) = \mu_1(E) - \mu_2(E)$$

is a signed measure.

3. Suppose

$$\int_X f^+ \, d\mu < \infty \quad \text{and} \quad \int_X f^- \, d\mu < \infty.$$

Then

$$\nu(E) = \int_E f \, d\mu$$

is a signed measure.

Note 21.2.1

Note that every measure is a signed measure. For emphasis, we shall also refer to our old definition of measures as the ordinary/regular/positive measure.

Notice that in the case of $\mathbb{R}$, we may write

$$E = (E \cap \mathbb{R}^+) \mathbin{\dot\cup} (E \cap \mathbb{R}^-),$$

where, for now, $\mathbb{R}^+ = \mathbb{R} \setminus (-\infty, 0)$ and $\mathbb{R}^- = \mathbb{R} \setminus [0, \infty)$. Then we may define

$$\begin{aligned} \nu(E) &= \int_{E \cap \mathbb{R}^+} f \, d\mu + \int_{E \cap \mathbb{R}^-} f \, d\mu \\ &= \int_{E \cap \mathbb{R}^+} f^+ \, d\mu - \int_{E \cap \mathbb{R}^-} f^- \, d\mu \\ &= \mu^+(E \cap \mathbb{R}^+) - \mu^-(E \cap \mathbb{R}^-). \end{aligned}$$

This motivates us to make the following definition.

Definition 34 (ν-positive and ν-negative)

*Let $(X, \mathfrak{M})$ be a measurable space, and ν a signed measure. We say that $E \in \mathfrak{M}$ is **ν-positive** if $\forall F \in \mathfrak{M}$ such that $F \subseteq E$, we have $\nu(F) \geq 0$. We say that $E \in \mathfrak{M}$ is **ν-negative** if $\forall F \in \mathfrak{M}$ such that $F \subseteq E$, we have $\nu(F) \leq 0$.*

Remark 21.2.2

It is important to note that the notion of ν-positivity and ν-negativity is not mutually exclusive, i.e. saying that a set A is "not ν-positive" does not mean that A is ν-negative, and vice versa.

22 Lecture 22 Nov 1st 2019

22.1 Hahn Decomposition

Theorem 56 (Hahn Decomposition Theorem)

Let ν be a signed measure on a measurable space $(X, \mathfrak{M})$. Then $\exists P \in \mathfrak{M}$ that is ν-positive and $N \in \mathfrak{M}$ that is ν-negative such that $X = P \mathbin{\dot\cup} N$. Furthermore, if $X = P' \mathbin{\dot\cup} N'$ for some other P' that is ν-positive and N' that is ν-negative, then $P \triangle P'$ and $N \triangle N'$ are ν-null.

Note 22.1.1
Recall that $P \triangle P'$ is called the symmetric difference of P and P', and it is defined as

$$P \triangle P' = (P \setminus P') \cup (P' \setminus P).$$

Homework (Homework 23)

Let ν be a signed measure. Let

$$E_1 \subseteq E_2 \subseteq \ldots, \text{ and } E = \bigcup_n E_n.$$

Then

$$\nu(E) = \lim_n \nu(E_n).$$

If $F_1 \supseteq F_2 \supseteq \ldots$ and $F = \bigcap_n F_n$, and $\nu(F_1) \neq \pm\infty$, then

$$\nu(F) = \lim_n \nu(F_n).$$

Lemma 57 (Lemma for Hahn Decomposition)

Suppose ν is a signed measure. Then

1. *P is ν-$(*)$ and $Q \subseteq P \implies Q$ is ν-$(*)$; and*
2. *P_n are ν-$(*)$, then $\bigcup_n P_n$ is ν-$(*)$,*

where ν-$(*)$ stands for ν-positive, ν-negative, and ν-null.

Proof (Lemma 57 for ν-positive case)

1. For any $R \in \mathfrak{M}$ such that $R \subseteq Q \subseteq P$, it follows that $\mu(R) \geq 0$ since P is ν-positive. Thus Q is ν-positive.
2. Let $A \in \mathfrak{M}$ such that $A \subseteq \bigcup_n P_n$. Notice that $A = \bigcup_n (P_n \cap A)$. In particular, $P_n \cap A \subseteq P_n$. Since P_n is ν-positive, we know $\nu(P_n \cap A) \geq 0$. Let

$$A_1 = A \cap P_1,\, A_2 = (A \cap (P_1 \cup P_2)) \setminus A_1, \ldots.$$

   In general
$$A_n = \left[A \cap \left(\bigcup_{i=1}^n P_i \right) \right] \setminus A_{n-1}.$$
   It is then clear that $A = \dot{\bigcup}_n A_n$. Thus $\nu(A) = \sum_{n=1}^\infty A_n$. Furthermore, since each $A_n \subseteq P_n$, it follows that $\nu(A_n) \geq 0$, and so $\nu(A) \geq 0$. □

Proof (Proof for Hahn Decomposition Theorem)

Let $E \in \mathfrak{M}$ such that $\nu(E) \neq \infty$. Let

$$M := \sup\{\nu(E) : E \text{ is } \nu\text{-positive }\}.$$

Note that set $\{\nu(E) : E \text{ is } \nu\text{-positive }\}$ is non-empty since $\emptyset$ is ν-positive, and so M is a valid value. We may then let $P_n \in \mathfrak{M}$ such that

$$\lim_n \nu(P_n) = M.$$

Let $P = \bigcup_n P_n$. By Lemma 57, we know that P is ν-positive. Furthermore, $\nu(P) \leq M$.

Now notice that $P = P_n \dot{\cup} (P \setminus P_n)$, and $P \setminus P_n \subseteq P$ and so by

Lemma 57 we have $\nu(P \setminus P_n) \geq 0$. It follows that

$$\nu(P) = \nu(P_n) + \nu(P \setminus P_n) \geq \nu(P_n)$$

for each n. Therefore

$$M = \sup \nu(P_n) = \lim_n \nu(P_n) \leq \nu(P).$$

Hence $\nu(P) = M$.

Let $N = X \setminus P$ so that $X = P \cup N$. WTS N is ν-negative. Firstly, consider $A \subseteq N$ and A is ν-positive. In particular, we must then have that $A \cup P$ is ν-positive. This means that

$$M \geq \nu(A \cup P) = \nu(A) + \nu(P) = \nu(A) + M, \tag{22.1}$$

which means that $\nu(A) = 0$. Furthermore, $\forall B \in \mathfrak{M}$ such that $B \subseteq A$, by the same reasoning as in Equation (22.1), we have that $\nu(B) = 0$. It follows that $\forall A \subseteq N$ that is ν-positive, A is ν-null.

Suppose to the contrary that N is not ν-positive. This means that $\forall A \subseteq N$, $\nu(A) > 0$. However, we showed that $\nu(A) = 0$. So A cannot be ν-positive. This implies that $\exists B_0 \in \mathfrak{M}$ with $B_0 \subseteq A$, we have $\nu(B_0) > 0$.

Let $C = A \setminus B_0$, so that $A = B_0 \cup C$. Then

$$0 < \nu(A) = \nu(B_0) + \nu(C),$$

and in particular

$$\nu(A) < \nu(C).$$

Inductively, let us perform the following. Due to our above argument about B_0, we may let

$$n_1 = \min \left\{ n : B \subseteq N, \, \nu(B) > \frac{1}{n} \right\}.$$

Let $A_1 \subseteq N$ such that $\nu(A_1) > \frac{1}{n_1}$. By the above argument about C, we know that $\exists C \subseteq A_1$ such that $\nu(C) > \nu(A_1)$. Let

$$n_2 = \min \left\{ n : C \subseteq A_1, \, \nu(C) > \frac{1}{n} + \nu(A_1) \right\}.$$

Then set $A_2 \subseteq A_1$ such that

$$\nu(A_2) > \nu(A_1) + \frac{1}{n_2} > \frac{1}{n_1} + \frac{1}{n_2}.$$

We continue inductively so that we construct as a decreasing sequence

$$\ldots \subseteq A_{k+1} \subseteq A_k \subseteq \ldots$$

of sets, such that

$$\nu(A_{k+1}) > \nu(A_k) + \frac{1}{n_{k+1}} > \frac{1}{n_1} + \ldots + \frac{1}{n_k} + \frac{1}{n_{k+1}}.$$

Let $A = \bigcap_k A_k$. Notice that since $\nu(A_1)$ is finite, by continuity from above,

$$\nu(A) = \lim_k \nu(A_k) \geq \sum_{k=1}^{\infty} \frac{1}{n_k}.$$

Need clarification □

23 Lecture 23 Nov 4th 2019

23.1 Hahn Decomposition (Continued)

Definition 35 (Hahn Decomposition)

Let ν be a signed measure, and $X = P \mathbin{\dot\cup} N$, where P is ν-positive and N is ν-negative. We call $P \mathbin{\dot\cup} N$ the ***Hahn decomposition*** *of ν.*

Remark 23.1.1

Given a Hahn decomposition, consider $E \subseteq X$. Then we may define $\nu_1(E) = \nu(E \cap P)$ such that ν_1 is a positive measure, and $\nu_2(E) = -\nu(E \cap N)$ such that ν_2 is a positive measure. Then

$$\nu(E) = \nu_1(E) - \nu_2(E).$$

Definition 36 (Mutually Singular)

Let $(X, \mathfrak{M})$ be a measurable space, with signed measures μ and ν. The measures μ and ν are said to be ***mutually singular****, of which we denote by $\mu \perp \nu$, when*

$$X = E \mathbin{\dot\cup} F \quad E, F \in \mathfrak{M}$$

such that E is μ-null and F is ν-null.

Example 23.1.1

From our remark above, we see that N is ν_1-null while P is ν_2-null. Thus $\nu_1 \perp \nu_2$.

Theorem 58 (Jordan Decomposition Theorem)

Let $(X, \mathfrak{M})$ be a measurable space with a signed measure ν. Then $\exists! \nu^+, \nu^-$ positive measures such that

$$\nu = \nu^+ - \nu^- \quad \textit{and} \quad \nu^+ \perp \nu^-.$$

Proof

(Existence) The proof for existence is exactly what we showed in the last remark. Let $X = P \cup N$ by the Hahn Decomposition Theorem with $\nu_1(E) = \nu(E \cap P)$ and $\nu_2 = -\nu(E \cap N)$, so that

$$\nu = \nu_1 - \nu_2 \quad \text{and} \quad \nu_1 \perp \nu_2.$$

(Uniqueness) Suppose $\nu(E) = \mu_1(E) - \mu_2(E)$ for some positive measures μ_1, μ_2 such that $\mu_1 \perp \mu_2$, so that

$$X = E \cup F$$

with F being μ_1-null and E being μ_2-null.

Let $A \subseteq E$. Then notice that

$$\nu(A) = \mu_1(A) - \mu_2(A) = \mu_1(A) \geq 0.$$

Thus E is ν-positive. Similarly, F is ν-negative. Hence $X = E \cup F$ is indeed another Hahn Decomposition on ν. By the Hahn Decomposition Theorem, we have that

$$P \triangle E,\, N \triangle F \text{ are } \nu\text{-null}.$$

Let $A \in \mathfrak{M}$. Since E and F are disjoint,

$$\mu_1(A) = \mu_1(A \cap E) + \mu_1(A \cap F) = \mu_1(A \cap E).$$

Note that

$$A \cap E = (A \cap E \cap P) \mathbin{\dot\cup} (A \cap (E \setminus P)).$$

Then

$$\begin{aligned} \mu_1(A \cap E) &= \nu(A \cap E) \\ &= \nu(A \cap E \cap P) + \nu(A \cap (E \setminus P)) \\ &= \nu(A \cap E \cap P) = \nu_1(A \cap E), \end{aligned}$$

where $\nu(A \cap (E \setminus P)) = 0$ since $P \triangle E$ is ν-null. Thus

$$\mu_1(A) = \mu_1(A \cap E) = \nu_1(A \cap E).$$

Also, notice that

$$A \cap P = (A \cap P \cap E) \mathbin{\dot\cup} (A \cap (P \setminus E)),$$

and so

$$\begin{aligned} \nu_1(A) &= \nu(A \cap P) \\ &= \nu(A \cap P \cap E) + \nu(A \cap (P \setminus E)) \\ &= \nu(A \cap E \cap P) = \nu_1(A \cap E) = \mu_1(A), \end{aligned}$$

since $P \setminus E \subseteq P \triangle E$ which is ν-null. Thus $\mu_1 = \nu_1$. Similarly, we can show that $\mu_2 = \nu_2$. □

Definition 37 (Positive and Negative Variation of a Signed Measure)

*Let $(X, \mathfrak{M})$ be a measurable space and ν a signed measure of the space. By the Jordan Decomposition Theorem, we may write $\nu = \nu^+ - \nu^-$ such that $\nu^+ \perp \nu^-$. We call ν^+ the **positive variation of ν** and ν^- the **negative variation of ν**.*

Definition 38 (Total Variation of a Signed Measure)

By the same assumption as the above, we call

$$|\nu| = \nu^+ + \nu^-$$

the ***total variation of*** ν.

Warning

Consider the measure

$$\nu(E) = \int_E f \, d\mu,$$

where μ *is any positive measure,* $E \subseteq X = P \cup N$. *Suppose* $f^+ = 0$ *on* N *and* $f^- = 0$ *on* P*, where* $f = f^+ - f^-$. *It is easy to see that*

$$\nu^+(E) = \int_E f^+ \, d\mu$$
$$\nu^-(E) = \int_E f^- \, d\mu.$$

Then the total variation is expressible as

$$|\nu|(E) = \int_E |f| \, d\mu.$$

However, recall that

$$|\nu(E)| = \left| \int_E f \, d\mu \right| \leq \int_E |f| = |\nu|(E).$$

Thus, we may have

$$|\nu(E)| < |\nu|(E).$$

23.2 *Radon-Nikodym Theorem and the Lebesgue Decomposition Theorem*

Definition 39 (Absolutely Continuous Measure)

Let $(X, \mathfrak{M})$ *be a measurable space. Let* ν *be a signed measure and* μ *a positive measure. We say that* ν *is* ***absolutely continuous with respect*** *to* μ, *of which we denote* $\nu \ll \mu$, *provided that*

$$\mu(E) = 0 \implies \nu(E) = 0$$

for all $E \subseteq X$.

Lemma 59 (Equivalent Definitions of a Absolutely Continuous Measure)

Let $(X, \mathfrak{M})$ *be a measurable space,* ν *a signed measure and* μ *a positive measure. TFAE:*

1. $\nu \ll \mu$;
2. $\nu^+ \ll \mu$ *and* $\nu^- \ll \mu$; *and*
3. $|\nu| \ll \mu$.

Proof

(1) $\implies$ (2) Write $\nu = \nu^+ - \nu^-$ by the Jordan Decomposition Theorem. Then for any $E \in \mathfrak{M}$ such that $\mu(E) = 0$, since $\nu(E) = 0$, we have

$$\nu^+(E) - \nu^-(E) = 0 \text{ and so } \nu^+(E) = \nu^-(E).$$

Now, note that $\mu(E) = 0 \implies \mu(E \cap P) = 0$ by subadditivity. It thus follows from assumption that $\nu^+(E) = \nu(E \cap P) = 0$ by the assumption that $\nu \ll \mu$.

(2) $\implies$ (3) Since $|\nu| = \nu^+ + \nu^-$, it follows that for any $E \in \mathfrak{M}$ such that $\mu(E) = 0$, we have $\nu^+(E) = 0 = \nu^-(E)$, and so

$$|\nu|(E) = \nu^+(E) + \nu^-(E) = 0.$$

(3) $\implies$ (1) Notice that for any $E \in \mathfrak{M}$, we have $|\nu(E)| \leq |\nu|(E)$, as noted in the last warning. Therefore, $\forall E \in \mathfrak{M}$ such that $\mu(E) = 0$, it follows that

$$|\nu(E)| \leq |\nu|(E) = 0,$$

and so it must be that $\nu(E) = 0$. □

Theorem 60 (Alternative Definition for Absolute Continuity of Measures)

Let ν be a finite signed measure, and μ a positive measure. Then $\nu \ll \mu \iff$

$$\forall \varepsilon > 0 \; \exists \delta > 0 \;\; \mu(E) < \delta \implies |\nu(E)| < \varepsilon.$$

Proof

$\implies$ Suppose that the ε-δ condition fails. Let $\varepsilon_0 > 0$ be the ε that fails. Consider $\delta_n = \frac{1}{2^n}$. Then there exists $P_n \in \mathfrak{M}$ such that

$$\mu(P_n) < \delta_n \wedge |\nu(P_n)| \geq \varepsilon_0.$$

Let $F_k = \bigcup_{n=k}^{\infty} P_n$ and $F = \bigcap_{k=1}^{\infty} F_k$. By subadditivity, for any k,

$$\mu(F_k) \leq \sum_{n=k}^{\infty} \mu(P_n) < \sum_{n=k}^{\infty} \frac{1}{2^n} = \frac{1}{2^{k-1}}$$

It follows from continuity from above that

$$\mu(F) = \lim_k \mu(F_k) = 0.$$

However,

$$\nu(F_k) \geq \nu(P_k) \geq \varepsilon_0$$

and ν is **finite**, $\nu(F_1) < \infty$, and so continuity from below indicates that

$$\nu(F) = \lim_k \nu(F_k) \geq \varepsilon_0.$$

Thus $\nu \not\ll \mu$. ($\impliedby$) Suppose that the ε-δ condition is true. In particular, we may consider a sequence $\varepsilon_n = \frac{1}{n} > 0$ so that $\exists \delta_n > 0$ such that

$$\mu(E) < \delta_n \implies |\nu(E)| < \frac{1}{n}$$

for any $E \in \mathfrak{M}$. Then in particular, if $E \in \mathfrak{M}$ such that $\mu(E) = 0$, then $\mu(E) < \delta_n$ for any n, and so $|\nu(E)| < \frac{1}{n}$ for any n. Therefore

$\nu(E) = 0$. $\square$

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24.1 Radon-Nikodym Theorem and the Lebesgue Decomposition Theorem (Continued)

Corollary 61 (On a special signed measure)

Let $f \in \mathcal{L}^1(\mu)$. Then $\forall \varepsilon > 0, \exists \delta > 0$ such that if $\mu(E) < \delta$, then $\left| \int_E f \, d\mu \right| < \varepsilon$.

Proof

Let $\nu(E) = \int_E f \, d\mu$. We know that ν is a finite signed measure. Notice that $\mu(E) = 0$ means that $\nu(E) = \int_E f \, d\mu = 0$. Hence $\nu \ll \mu$. Therefore, by Theorem 60, the statement holds. □

Lemma 62 (Relationship Between Two Finite Positive Measure)

Let μ and ν be finite positive measures on the measurable space $(X, \mathfrak{M})$. Then either $\mu \perp \nu$, or [1] $\exists \varepsilon > 0$ and $\exists E \in \mathfrak{M}$ with $\mu(E) > 0$ such that $\forall F \subseteq E$, we have

$$\nu(F) \geq \varepsilon \mu(F),$$

i.e. $\nu - \varepsilon \mu$ is a positive measure on $(E, \mathfrak{M}_E)$.

[1] This is an exclusive or statement.

Proof

For each $n \geq 1$, consider the signed measure $\nu - \frac{1}{n}\mu$, which is finite. By the Hahn Decomposition Theorem, let P_n and N_n be such that $X = P_n \bigcup N_n$ with P_n being $\left(\nu - \frac{1}{n}\mu\right)$-positive, and N_n being $\left(\nu - \frac{1}{n}\mu\right)$-negative.

Let $P = \bigcup_n P_n$ and $N = \bigcap N_n$. Then $X = P \bigcup N$. Since $N \subseteq N_n$ for all n, we have that

$$\left(\nu - \frac{1}{n}\mu\right)(N) \leq 0 \quad \forall n.$$

Thus

$$0 \leq \nu(N) \leq \frac{1}{n}\mu(N) \quad \forall n.$$

Hence $\nu(N) = 0$, i.e. N is ν-null.

Now if $\mu(P) = 0$, then $\mu \perp \nu$ and we are done. WMA $\mu(P) \neq 0$. Since $P = \bigcup_n P_n$, we know that $\exists n_0$ such that $\mu(P_{n_0}) \neq 0$. However, P_{n_0} is $\left(\nu - \frac{1}{n}\mu\right)$-positive, i.e.

$$\forall A \subseteq P_{n_0} \quad \nu(A) - \frac{1}{n}\mu(A) \geq 0.$$

Then in this case, we simply need to pick $\varepsilon = \frac{1}{n_0}$, and our job is done. □

Theorem 63 (The Lebesgue Decomposition Theorem and The Radon-Nikodym Theorem)

*Let $(X, \mathfrak{M})$ be a measurable space, and μ and ν be σ-finite positive measures. Then, there exists positive measures λ and ρ such that $\mu \perp \lambda$, $\rho \ll \mu$ and $\nu = \rho + \lambda$ (**Lebesgue Decomposition**). Moreover, $\exists f : X \to [0, \infty]$ that is $\mathfrak{M}$-measurable such that*

$$\rho(E) = \int_E f \, d\mu$$

*(**Radon-Nikodym**).*

This is a constructive proof. It tells us exactly how to create λ and ρ.

Proof

Case 1: μ and ν are both finite Let

$$\mathcal{F} := \left\{ f : X \to [0, \infty] \mid f \text{ is } \mathfrak{M}\text{-measurable}, \int_E f \, d\mu \leq \nu(E) \ \ \forall E \in \mathfrak{M} \right\}.$$

Note that $\mathcal{F} \neq \emptyset$ since $0 \in \mathcal{F}$.

$\mathcal{F}$ is closed under max function Let $f, h \in \mathcal{F}$ and $g = \max\{f, h\}$. Let

$$A := \{x : g(x) = f(x)\}.$$

Then for $x \in A^C$, we have $g(x) = h(x)$. Note that g is $\mathfrak{M}$-measurable. WTS $g \in \mathcal{F}$. Observe that

$$\begin{aligned} \int_E g \, d\mu &= \int_{E \cap A} g \, d\mu + \int_{E \cap A^C} g \, d\mu \\ &= \int_{E \cap A} f \, d\mu + \int_{E \cap A^C} h \, d\mu \\ &\leq \nu(E \cap A) + \nu(E \cap A^C) = \nu(E). \end{aligned}$$

Hence indeed $g \in \mathcal{F}$. $\dashv$

$\mathcal{F}$ contains limits of its sequences Let

$$a := \sup \left\{ \int_X f \, d\mu : f \in \mathcal{F} \right\}.$$

Let $\{f_n\} \subseteq \mathcal{F}$ be a sequence of functions such that

$$\int_X f_n \, d\mu \to a.$$

Let $g_1 = f_1$, $g_2 = \max\{f_1, f_2\}, \ldots, g_n = \max_n\{f_1, \ldots, f_n\}, \ldots$. Let $f = \lim_n g_n$. Then $g_n \nearrow f$. By the last claim, note that each $g_n \in \mathcal{F}$. WTS $f \in \mathcal{F}$.

Let $E \in \mathfrak{M}$. By the ★ Monotone Convergence Theorem (MCT),

$$\begin{aligned} \int_E f \, d\mu &= \lim_n \int_E g_n \, d\mu \\ &\leq \lim_n \nu(E) = \nu(E). \end{aligned}$$

It follows that $f \in \mathcal{F}$ and in particular

$$\int_X f \, d\mu = a. \dashv$$

Let $\rho(E) = \int_E f \, d\mu$. Then it is clear that $\rho \ll \mu$. Let $\lambda(E) = \nu(E) - \rho(E) \geq 0$. Then λ is a positive measure. Furthermore, we have that $\nu = \lambda + \rho$.

$\mu \perp \lambda$ Suppose not. Then by Lemma 62, we know $\exists \varepsilon > 0$ and $\exists E \in \mathfrak{M}$ with $\mu(E) > 0$ such that $\forall F \subseteq E$, $\lambda(F) \geq \varepsilon \mu(F)$. Then, $\forall A \in \mathfrak{M}$, we have

$$\lambda(A) \geq \lambda(A \cap E) \geq \varepsilon \mu(A \cap E).$$

However,

$$\lambda(A) = \nu(A) - \int_A f \, d\mu \geq \varepsilon \mu(A \cap E) = \varepsilon \int_A \chi_E \, d\mu.$$

This means that

$$\nu(A) \geq \int_A (f - \varepsilon \chi_E) \, d\mu.$$

Hence $f - \varepsilon \chi_E \in \mathcal{F}$. Thus

$$a = \int_X f \, d\mu \leq \int_X (f + \varepsilon \chi_E) \, d\mu \leq a.$$

Notice that $\int_X (f + \varepsilon \chi_E) \, d\mu = a + \varepsilon \mu(E)$, and so

$$a \leq a + \varepsilon \mu(E) \leq a.$$

Thus we must have $\varepsilon \mu(E) = 0$, but that is impossible since $\varepsilon > 0$ and $\mu(E) > 0$, a contradiction. Hence we must have $\mu \perp \lambda$.

Case 2: ν and μ are both σ-finite [2] Since μ is σ-finite, wma $X = \dot{\bigcup}_i A_i$ such that $\mu(A_i) < \infty$. Since ν is σ-finite, wma $X = \dot{\bigcup}_i B_i$ such that $\nu(B_i) < \infty$. Then we may let $X = \dot{\bigcup}_{i,j} (A_i \cap B_j)$, and

$$\mathfrak{M}_{i,j} \coloneqq \{E \cap A_i \cap B_j : E \in \mathfrak{M}\}$$

would be a σ-algebra of subsets of X.

[2] We shall strict the finite pieces together and then try paying attention to some of the technical details.

Let us define

$$\mu_{i,j}(Y) = \mu(Y) \quad \forall Y \in \mathfrak{M}_{i,j} \quad \text{and} \quad \nu_{i,j}(Y) = \nu(Y) \quad \forall Y \in \mathfrak{M}_{i,j}.$$

It is clear that all of $\mu_{i,j}$ and $\nu_{i,j}$ are finite. By *Case 1*, $\exists \lambda_{i,j}, \rho_{i,j}$ measures and $\exists f_{i,j} : A_i \cap B_j \to \mathbb{R}_e$ such that

$$\lambda_{i,j} \perp \mu, \quad \rho_{i,j}(Y) = \int_Y f_{i,j} \, d\mu_{i,j} \quad \text{and} \quad \nu_{i,j} = \lambda_{i,j} + \rho_{i,j}.$$

Therefore $A_i \cap B_j = W_{ij} \cup Z_{ij}$ such that

$$\lambda_{i,j}(Z_{ij}) = 0 \quad \text{and} \quad \mu_{i,j}(W_{ij}) = 0.$$

Define $f : X \to \mathbb{R}$ by $f(x) = f_{i,j}(x)$ when $x \in A_i \cap B_j$. Notice that

$$f^{-1}(a, \infty] = \bigcup_{i,j} f_{i,j}(a, \infty] \in \mathfrak{M},$$

and thus f is $\mathfrak{M}$-measurable.

Define $\lambda(E) = \sum_{i,j} \lambda_{i,j}(E \cap A_i \cap B_j)$. It is easy to **check** that λ is indeed a measure.

Let $W = \dot{\bigcup}_{i,j} W_{ij}$ and $Z = \dot{\bigcup}_{i,j} Z_{ij}$. Then

$$\lambda(Z) = \sum_{i,j} \lambda_{i,j}(Z_{ij}) = 0,$$

and

$$\mu(W) = \sum_{i,j} \mu(W_{ij}) = \sum_{i,j} \mu_{i,j}(W_{ij}) = 0.$$

Thus $\lambda \perp \mu$.

Let

$$\rho(E) = \int_E f \, d\mu = \sum_{i,j} \int_{E \cap A_i \cap B_j} f_{i,j} \, d\mu = \sum_{i,j} \rho_{i,j}(E \cap A_i \cap B_j).$$

Check that

$$\begin{aligned} \lambda(E) + \rho(E) &= \sum_{i,j} \lambda_{i,j}(E \cap A_i \cap B_j) + \sum_{i,j} \rho_{i,j}(E \cap A_i \cap B_j) \\ &= \sum_{i,j} \nu_{i,j}(E \cap A_i \cap B_j) = \nu(E). \end{aligned}$$

This completes the proof. □

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25.1 Radon-Nikodym Theorem and The Lebesgue Decomposition Theorem (Continued 2)

Corollary 64 (A Corollary to Radon-Nikodym)

Suppose that $(X, \mathfrak{M})$ is a measurable space and μ, ν are σ-finite measures on the space, such that $\nu \ll \mu$. Then $\exists f : X \to [0, \infty]$ $\mathfrak{M}$-measurable such that

$$\nu(E) = \int_E f \, d\mu.$$

Proof

By the Radon-Nikodym Theorem, we can find positive measures λ, ρ such that $\nu = \lambda + \rho$, $\lambda \perp \mu$ and $\rho(E) = \int_E f \, d\mu$ for some $f : X \to [0, \infty]$. We may then let $X = A \cup B$ such that $\lambda(A) = 0$ and $\mu(B) = 0$. Since $\nu \ll \mu$, we have $\nu(B) = 0$.

Let $E \in \mathfrak{M}$. Then

$$\begin{aligned}
\nu(E) = \nu(E \cap A) + \nu(E \cap B) &= \nu(E \cap A) \\
&= \lambda(E \cap A) + \int_{E \cap A} f \, d\mu \\
&= \int_{E \cap A} f \, d\mu + \int_{E \cap B} f \, d\mu \quad \because \rho(E \cap B) = 0 \text{ as } \rho \ll \mu \\
&= \int_E f \, d\mu.
\end{aligned}$$

□

Note 25.1.1

From here on, we shall denote the Lebesgue measure by m, instead of μ, which we shall reserve for an arbitrary μ.

Let us take a short detour into derivatives.

Example 25.1.1

1. Let $F : \mathbb{R} \to \mathbb{R}$ such that

$$F(x) = \begin{cases} 0 & x \leq 0 \\ x^2 & 0 < x < 1 \\ x^2 + 3 & x \geq 1 \end{cases}.$$

Then F is increasing and right continuous. Let

$$F_1(x) = \begin{cases} 0 & x \leq 0 \\ x^2 & x > 0 \end{cases} \quad F_2(x) = \begin{cases} 0 & x < 1 \\ 3 & x \geq 1 \end{cases}.$$

Then $F = F_1 + F_2$. We may thus define $\mu_F = \mu_{F_1} + \mu_{F_2}$ for $\mathfrak{B}(\mathbb{R})$. From our understanding (from a much earlier example), we know that $\mu_{F_2} = 3 \cdot \chi_{\{1\}}$. Let

$$f(x) = \begin{cases} 0 & x \leq 0 \\ 2x & x > 0 \end{cases}.$$

Notice that f is the **derivative** of F_1. By the Agreement of Riemann Integration and Lebesgue Integration for Bounded Functions, we know

$$F_1(b) - F_1(a) = \int_{[a,b]} f \, dm.$$

Since m is the Lebesgue measure, and by Theorem 12, we know

$$\int_{[a,b]} f \, dm = \int_{(a,b]} f \, dm = \mu_{F_1}(a, b] = F_1(b) - F_1(a).$$

Thus, since we may write any subset of $\mathbb{R}$ as a disjoint union of intervals, it follows that

$$\mu_{F_1}(B) = \int_B f \, dm.$$

Notice that μ_{F_1} and μ_{F_2} are one of the decompositions given in The Lebesgue Decomposition Theorem and The Radon-Nikodym Theorem, where $\nu = \mu_F$, $\lambda = \mu_{F_2}$, $\rho = \mu_{F_1}$, and $\mu = m$, where we note that $\mu_{F_2} \perp m$,

$$\mu_{F_1}(B) = \int_B f \, dm,$$

and clearly $\mu_{F_1} \ll m$.

2. Recall the Cantor Function, which here we shall call it F_C. Consider the function $F : \mathbb{R} \to \mathbb{R}$ such that

$$F(x) = \begin{cases} 0 & x \leq 0 \\ F_C(x) & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}.$$

Then F is continuous and increasing. Furthermore, $F'(x) = 0$ whenever $x \notin C$, where C is the Cantor set. Consider the function μ_F constructed by the means of Theorem 12.

Now, notice that we may write $\mathbb{R} = C \cup C^C$. Recall that we showed that $\mu_F(C^C) = 0$ while $\mu_F(C) = 1$. Furthermore, under Lebesgue's measure, $m(C) = 0$. In this case, we may consider $\lambda = \mu_F$ and $\mu = 0$, and $\rho = 0$ to fit the framework of Theorem 63.

Note 25.1.2

We know that give σ-finite measures ν and μ, where $\nu \ll \mu$, by Corollary 64, $\exists f \geq 0$ such that $\nu(E) = \int_E f \, d\mu$. Recall from the Midterm problem that

$$\int_E h \, d\nu = \int_E h \cdot f \, d\mu.$$

We may thus look at $d\nu$ as something like $f \, d\mu$, which then f is something like $\frac{d\nu}{d\mu}$.

Remark 25.1.1

Given σ-finite measures ν and μ, with $\nu \ll \mu$, if we suppose $f, h \geq 0$, and

$$\nu(E) = \int_E f \, d\mu = \int_E h \, d\mu,$$

we know that $f = h$ a.e. with respect to μ.

Definition 40 (Derivative)

Let ν and μ be σ-finite measures with $\nu \ll \mu$. If $\nu(E) = \int_E f \, d\mu$, we say that

$$\frac{d\nu}{d\mu} = f \text{ a.e. with respect to } \mu.$$

Note 25.1.3 (Some formulae)

1. *Given $\nu_1, \ldots, \nu_n, \mu$ all σ-finite, with $\nu_i \ll \mu$ for all i, we know that by* *Corollary 64*, *$\exists f_1, \ldots, f_n$ such that*

   $$\nu_i(E) = \int_E f_i \, d\mu, \quad \text{for each } i.$$

   Let

   $$\nu(E) := \sum_i \nu_i(E) = \int_E (f_1 + \ldots + f_n) \, d\mu.$$

   Then

   $$\frac{d\nu}{d\mu} = \frac{d\nu_1}{d\mu} + \ldots + \frac{d\nu_n}{d\mu}$$

   a.e. with respect to μ.

2. *(**Chain rule**) Suppose $\nu \ll \lambda \ll \mu$. Note $\nu \ll \mu$. Then*

   $$\lambda(E) = \int_E g \, d\mu \quad \text{and} \quad \nu(E) = \int_E f \, d\lambda$$

   for some functions f, g. By our Midterm problem, we have

   $$\nu(E) = \int_E f \cdot g \, d\mu.$$

   Then

   $$\frac{d\nu}{d\mu} = f \cdot g, \; f = \frac{d\nu}{d\lambda}, \text{ and } g = \frac{d\lambda}{d\mu}.$$

Hence

$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda} \cdot \frac{d\lambda}{d\mu}.$$

3. (*Change of variables*) *Suppose* $\nu \ll \mu \ll \nu$. *Then there exists functions* f, g *such that*

$$\nu(E) = \int_E f \, d\mu \quad \text{and} \quad \mu(E) = \int_E g \, d\nu.$$

Then we say

$$f = \frac{d\nu}{d\mu} \quad \text{and} \quad g = \frac{d\mu}{d\nu}.$$

By the Midterm problem, [1]

$$\nu(E) = \int_E f \cdot g \, d\nu \quad \forall E.$$

Thus

$$f \cdot g = 1 \text{ a.e. wrt } \nu \quad \text{and} \quad g \cdot f = 1 \text{ a.e. wrt } \mu.$$

Thus means that the sets

$$A := \{x : f(x) = 0\} \quad \text{and} \quad B := \{x : g(x) = 0\}$$

both have measure zero. Thus

$$\left(\frac{d\nu}{d\mu}\right)^{-1} = \frac{d\mu}{d\nu}$$

wherever they are well-defined.

[1] This is the worst measure I've seen.

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26.1 Non-decreasing Functions

Note 26.1.1

For sanity (of keeping up with notes for the rest of the course), we shall refer to non-decreasing functions are increasing functions. We shall call increasing functions are strictly increasing functions.

Notation

Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function. We shall denote

$$F(x-) := \lim_{y \to x^-} F(y) = \sup\{F(y) : y < x\},$$

and

$$F(x+) = \lim_{y \to x^+} F(y) = \inf\{F(y) : y > x\}.$$

Proposition 65 (Properties of $F(x+)$ and $F(x-)$)

Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function.

1. $x_1 < x_2 \implies F(x_1+) \leq F(x_2-)$.
2. $F(x-)$ *and* $F(x+)$ *are both increasing.*

3. *$F(x-)$ is left continuous, i.e.*

$$F(x-) = \lim_{y \to x^-} F(y-).$$

4. *$F(x+)$ is right continuous, i.e.*

$$F(x+) = \lim_{y \to x^-} F(y+).$$

5. *The set of points where F is not continuous is countable.*

6. *F is $\mathfrak{B}(\mathbb{R})$-measurable.*

Proof

1. Let $x_3 \in \mathbb{R}$ be such that $x_1 < x_3 < x_2$. It is clear that

$$F(x_1+) = \inf\{F(y) : y > x_1\} \leq F(x_3)$$

and

$$F(x_3) \leq \sup\{F(y) : y < x_2\} = F(x_2-).$$

2. Let $x_1 < x_2$. Then

$$F(x_1-) = \sup\{F(y) : y < x_1\} \leq \sup\{F(y) : y < x_2\} = F(x_2-)$$

because the latter set allows for more choice for a supremum. For the other case,

$$F(x_1+) = \inf\{F(y) : y > x_1\} \leq \inf\{F(y) : y > x_2\} = F(x_2+)$$

because the former set allows for more choice for an infimum.

3. Let $\varepsilon > 0$. For any $x \in \mathbb{R}$, since

$$F(x-) = \sup\{F(y) : y < x\},$$

we know that $\exists y < x$ such that $\left|F(x-) - F(y)\right| < \frac{\varepsilon}{3}$. Furthermore, for each $y < x$, since

$$F(y-) = \sup\{F(z) : z < y\},$$

we can find $z < y < x$ such that $\left|F(y-) - F(z)\right| < \frac{\varepsilon}{3}$.

Suppose to the contrary that $\left|F(y-) - F(y)\right| \geq \frac{\varepsilon}{3}$. This means that

$$\sup\{F(z) : z < y\} - F(y) > 0,$$

and so there must exist some $z < y$ such that $F(z) > F(y)$, but that contradicts the fact that F is increasing. Thus $\left|F(y-) - F(y)\right| < \frac{\varepsilon}{3}$.

It follows that $\exists z < x$ such that

$$|F(x-) - F(z)| \leq \left|F(x-) - F(y)\right| + \left|F(y) - F(y-)\right| + \left|F(y-) - F(z)\right| < \varepsilon.$$

4. The proof for (4) is similar to that of (3).

5. Let
$$B := \{x : F \text{ is discontinuous at } x\}.$$

   Note that F being continuous at a point x means that

$$F(x-) = F(x) = F(x+).$$

   Since we always have $F(x-) \leq F(x+)$, if F is discontinuous at x, then we must have $F(x-) < F(x+)$.

   If F has only one point of discontinuity, then our job is done.

   Let x_1 and x_2 be 2 points of discontinuity of F on $\mathbb{R}$. WLOG, suppose $x_1 < x_2$. Then by part (1), we have

$$F(x_1-) < F(x_1+) \leq F(x_2-) < F(x_2+).$$

   This means that if we consider $r_{x_1}, r_{x_2} \in \mathbb{Q}$ such that

$$F(x_1-) < r_{x_1} < F(x_1+) \leq F(x_2-) < r_{x_2} < F(x_2+),$$

   we know that we can clearly distinguish r_{x_1} and r_{x_2}. If we then consider the map

$$\begin{aligned} f : B &\to \mathbb{Q} \\ x &\mapsto r_x, \end{aligned}$$

f must be one-to-one, and so B is at most countable.

6. Let $a \in \mathbb{R}$, and consider $A = (a, \infty)$. WTS $F^{-1}(A) \in \mathfrak{B}(\mathbb{R})$. Note that $x \in F^{-1}(A) \iff F(x) > a$. Since F is increasing, $\forall y > x$, we know
$$a < F(x) \leq F(y),$$
which implies that $y \in F^{-1}(A)$ as well. It follows that
$$F^{-1}(A) = \begin{cases} \emptyset \\ [b, \infty) \\ (b, \infty) \\ \mathbb{R} \end{cases},$$
where $b \in \mathbb{R}$ can be checked, and we note that each of the possible forms of $F^{-1}(A)$ is a Borel set. □

26.2 *Vitali Covering Lemma*

Notation

We shall use the symbol $\mathfrak{I}$ to denote a collection of intervals, which are possibly open, closed, or half-open, but cannot be singletons.

Definition 41 (Vitali Covering)

Let $E \subseteq \mathbb{R}$. We say that $\mathfrak{I}$ is a ***Vitali covering*** *of E if $\forall \varepsilon > 0$, $\forall x \in E$, $\exists I \in \mathfrak{I}$ such that $x \in I$ and $\ell(I) < \varepsilon$.*

Remark 26.2.1

In words, a Vitali covering is defined such that for every positive ε, every point x in E can be covered by an interval of length less than ε. So a Vitali covering is a special kind of ***cover*** *for E, and also a special kind of* ***ε-net****.*

Theorem 66 (Vitali Covering Lemma)

Suppose $E \subseteq \mathbb{R}$ and $m^*(E) < \infty$, where we note that m^* is the Lebesgue outer measure. Suppose $\mathfrak{I}$ is a Vitali covering of E. Let $\varepsilon > 0$. Then $\exists\{I_1, \ldots, I_n\} \subseteq \mathfrak{I}$ a disjoint collection

$$m^*\left(E \setminus \bigcup_{n=1}^{N} I_n\right) < \varepsilon.$$

Proof

First, notice that if we can find a Vitali covering of E where each interval is closed, such that the above statement holds, then we can also find intervals of other forms that work. Thus WLOG, assume every $I \in \mathfrak{I}$ is closed.

Will come back $\square$

27 Lecture 27 Nov 13th 2019

27.1 Vitali Covering Lemma (Continued)

Definition 42 (D^+, D^-, D_+, D_-)

Let $f : (a, b) \to \mathbb{R}$. We define

$$\begin{aligned} D^+ f(x) &:= \limsup_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \\ D^- f(x) &:= \limsup_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \\ &= \limsup_{h \to 0^+} \frac{f(x) - f(x-h)}{h} \\ D_+ f(x) &:= \liminf_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \\ D_- f(x) &:= \liminf_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \\ &= \liminf_{h \to 0^+} \frac{f(x) - f(x-h)}{h} \end{aligned}$$

Remark 27.1.1

1. *Since all the above are defined using* $\liminf$ *and* $\limsup$*, we know that these values always exists.*
2. *However, they are not always equal, since derivatives don't always exist. In other words, if all 4 are equal, we know that $f'(x)$ exists and is equal to the common value.*

Theorem 67 (Partial Fundamental Theorem of Calculus for Increasing Functions)

Let $f : [a, b] \to \mathbb{R}$ increasing, then $f'(x)$ exists for almost all x, $f'(x)$ is Lebesgue integrable, and

$$\int_a^b f'(x)\, dm \leq f(b) - f(a).$$

Example 27.1.1

Consider the Cantor Function F, where we recall

$$F(0) = 0,\, F(1) = 1, \text{ and } F'(x) = 0 \;\; \forall x \notin C,$$

and F is increasing. Notice that

$$0 = \int_0^1 F'\, dm < f(1) - f(0) = 1.$$

Proof (Outline of Theorem 67)

We'll come back □

Remark 27.1.2

From Example 27.1.1, we know that we cannot have the Fundamental Theorem of Calculus for general measures.

However, we are not going down without a fight. We shall now look into the class of functions that satisfy the Fundamental Theorem of Calculus for general measures.

27.2 Functions of Bounded Variation

Definition 43 (Functions of Bounded Variation)

Let $F : [a, b] \to \mathbb{C}$. [1] *We say that F is of* ***bounded variation*** *if*

[1] Note that we use $\mathbb{C}$ here, so that we allow for the function to map to both complex and also purely real values.

$$V_F[a,b] := \sup_{P[a,b]} \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : x_i \in P[a,b] \right\} < \infty,$$

where $P[a,b]$ *is a partition of* $[a,b]$. *We also write that* $F \in \mathrm{BV}[a,b]$. *We call* $V_F[a,b]$ *the* ***variation*** *of* F *on* $[a,b]$.

We may extend $[a,b]$ *to be the entire real line and consider, instead,*

$$V_F(\mathbb{R}) := \sup_{P(\mathbb{R})} \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : x_i \in P(\mathbb{R}) \right\},$$

where in this case, our partition $P(\mathbb{R})$ *can be*

$$P = \{x_0 < x_1 < \ldots < x_n\},$$

where x_0 *and* x_n *are arbitrary points in* $\mathbb{R}$.

28 Lecture 28 Nov 15th 2019

28.1 Functions of Bounded Variation (Continued)

Definition 44 (Total Variation)

*Let F be a function of bounded variation. The **total variation** of F is defined as*

$$T_F(x) := \sup_{\textit{partition}} \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : x_0 < x_1 < \ldots < x_n = x \right\}.$$

Remark 28.1.1

1. *T_F is increasing (non-decreasing).*
2. *$\forall x \in \mathbb{R}$, notice that*

$$0 \leq T_F(x) \leq V_F(\mathbb{R}).$$

Example 28.1.1

1. Let $F : [a, b] \to \mathbb{R}$ be increasing, with an arbitrary partition

$$a = x_0 < x_1 < \ldots < x_n = b.$$

Then

$$\begin{aligned} \textstyle\sum_{j=1}^{n} \left|F(x_j) - F(x_{j-1})\right| &= \textstyle\sum_{j=1}^{n} F(x_j) - F(x_{j-1}) \quad \because F \text{ is increasing} \\ &= F(x_n) - F(x_0) \\ &= F(b) - F(a) < \infty. \end{aligned}$$

Telescope

Thus $F \in \mathrm{BV}[a, b]$. Furthermore

$$V_F[a, b] = F(b) - F(a).$$

2. Consider the function $F(x) = x$. Then it's clear that $F \in \mathrm{BV}[a, b]$. However, $F \notin \mathrm{BV}(\mathbb{R})$, since $F(x) \to \infty$ as $x \to \infty$ and so $V_F(\mathbb{R}) = \infty$.

3. Let $F : [a, b] \to \mathbb{C} \in \mathrm{BV}[a, b]$. Let

$$\begin{aligned} \tilde{F} &: \mathbb{R} \to \mathbb{C} \\ x &\mapsto \begin{cases} F(b) & x \geq b \\ F(x) & a < x < b \\ F(a) & x \leq a \end{cases}. \end{aligned}$$

Thus $\tilde{F} \in \mathrm{BV}[a, b]$ as well, and

$$T_F[a, b] = T_{\tilde{F}}(\mathbb{R}).$$

This means that we can focus on functions that map from the whole real line.

4. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and bounded. Then

$$\lim_{x \to \infty} = \sup\{F(x) : x \in \mathbb{R}\} =: F(+\infty) < \infty$$

and

$$\lim_{x->-\infty} = \inf\{F(x) : x \in \mathbb{R}\} =: F(-\infty) < \infty.$$

Then $F \in \mathrm{BV}(\mathbb{R})$ and

$$T_F(\mathbb{R}) = F(+\infty) - F(-\infty).$$

5. Suppose $h \in \mathcal{L}^1(\mathbb{R}, m)$ Let

$$F(x) = \int_{(-infty,x)} h \, dm.$$

Then $F \in \mathrm{BV}(\mathbb{R})$, and

$$T_F(\mathbb{R}) \leq \int_{\mathbb{R}} |h| \, dm.$$

Proof

Let $x_0 < x_1 < \ldots < x_n$. Notice that

$$\begin{aligned} \left|F(x_j) - F(x_{j-1})\right| &= \left| \int_{(-\infty, x_j)} h \, dm - \int_{(-\infty, x_{j-1})} h \, dm \right| \\ &= \left| \int_{(x_{j-1}, x_j)} h \, dm \right| \\ &\leq \int_{(x_{j-1}, x_j)} \lvert h \rvert \, dm, \end{aligned}$$

for each j. Thus

$$\begin{aligned} \sum_{j=1}^{n} \left|F(x_j) - F(x_{j-1})\right| &\leq \int_{(x_0, x_n)} \lvert h \rvert \, dm \\ &\leq \int_{\mathbb{R}} \lvert h \rvert \, dm. \end{aligned}$$

It is then clear that

$$T_F(\mathbb{R}) \leq \int_{\mathbb{R}} \lvert h \rvert \, dm.$$

□

Lemma 68 (Total Variation $\pm F$)

Let $F : \mathbb{R} \to \mathbb{R} \in \mathrm{BV}(\mathbb{R})$. Then $T_F + F$ and $T_F - F$ are both increasing functions and bounded.

Proof

We'll come back

□

Note 28.1.1 (Jordan Decomposition of F)

For $F \in \mathrm{BV}(\mathbb{R})$, we may write

$$F = \left(\frac{T_F + F}{2} \right) - \left(\frac{T_F - F}{2} \right),$$

which is a difference of 2 *increasing bounded functions.*

Theorem 69 (Properties of Functions of Bounded Variation)

1. $\mathrm{BV}(\mathbb{R})$ *is a vector space.*
2. $$\begin{aligned} F \in \mathrm{BV}(\mathbb{R}) &\iff \overline{F} \in \mathrm{BV}(\mathbb{R}) \\ &\iff \Re(F), \Im(F) \in \mathrm{BV}(\mathbb{R}), \end{aligned}$$
   where $\overline{F}$ *is the complex conjugate of* F*.*
3. *Let* $F : \mathbb{R} \to \mathbb{R}$*. Then*
   $$F \in \mathrm{BV}(\mathbb{R}) \iff F = H_1 - H_2,$$
   where H_1, H_2 *are increasing and bounded.*
4. $$\begin{aligned} &F \in \mathrm{BV}(\mathbb{R}) \\ \implies &\lim_{y \to x^+} F(y), \lim_{y \to x^-} F(y), \lim_{y \to \infty} F(y), \lim_{y \to -\infty} F(y) \end{aligned}$$
   all exists.
5. *Let* $F \in \mathrm{BV}(\mathbb{R})$*. Then the set of points of discontinuity of* F *is at most countable.*
6. *Let* $F : \mathbb{R} \to \mathbb{R} \in \mathrm{BV}(\mathbb{R})$*. Then* $F'(x)$ *exists a.e.*

Proof

We'll come back □

Definition 45 (Absolutely Continuous Functions)

A function $f : \mathbb{R} \to \mathbb{C}$ *is said to be* ***absolutely continuous*** *if* $\forall \varepsilon > 0$*,*

$\exists \delta > 0$ such that $\forall (a_1, b_1), \ldots, (a_n, , b_n)$ that are disjoint, if $\sum_{j=1}^{n} (b_j - a_j) < \delta$, then

$$\sum_{j=1}^{n} \left| F(b_j) - F(a_j) \right| < \varepsilon.$$

We write that $f \in \mathrm{AC}(\mathbb{R})$.

Remark 28.1.2

Notice if there is only one of (a_i, b_i) that works, then the above definition is simply the definition of uniform continuity. Thus, since this ε works for all the intervals, in a sense we may think of absolutely continuous functions as "uniformly uniform continuous functions".

29 Lecture 29 Nov 18th 2019

29.1 Functions of Bounded Variation (Continued 2)

Proposition 70 (Measure Constructed From A Bounded, Absolutely Continuous, Increasing, Right-Continuous Function)

Let $H : \mathbb{R} \to \mathbb{R}$ be a bounded, right-continuous increasing function. Then

$$\mu_H \ll m \iff H \in \mathrm{AC}(\mathbb{R}).$$

Proof

We'll come back $\square$

Lemma 71 (Absolutely Continuous Functions are of Bounded Variation)

Let $F : [a, b] \to \mathbb{R}$ be absolutely continuous. Then $F \in \mathrm{BV}[a, b]$.

Proof

We'll come back $\square$

Lemma 72 (Total Variation of Absolutely Continuous Functions are Absolutely Continuous)

Let $F : [a, b] \to \mathbb{R}$ be absolutely continuous. Then $T_F(x)$ is also absolutely continuous.

Proof

We'll come back $\square$

Lemma 73 (Building Block for Fundamental Theorem of Calculus for General Measures)

Let f be a bounded measurable function, and $a \in \mathbb{R}$. Let

$$F(x) := \int_{(a,x)} f \, dm.$$

Then $F'(x) = f(x)$ a.e.

Proof

We'll come back $\square$

30 Lecture 30 Nov 20th 2019

30.1 Functions of Bounded Variation (Continued 3)

Theorem 74 (Anti-Derivative of Bounded Integrable Functions)

Let $f \in \mathcal{L}^1([a,b], m)$. Let

$$F(x) \coloneqq \int_{(a,x)} f \, dm.$$

Then F' exists a.e. and $F'(x) = f(x)$ a.e.

Proof

We'll come back $\square$

Theorem 75 (Fundamental Theorem of Calculus (Lebesgue Version))

Let $F : [a,b] \to \mathbb{R}$. TFAE:

1. *F is absolutely continuous.*
2. *$\exists f$ integrable such that*

$$F(x) = F(a) + \int_{(a,x)} f \, dm \quad \forall a \leq x \leq b.$$

3. *$F'(x)$ exists a.e., F' is integrable and*

$$F(x) = F(a) + \int_{(a,x)} F' \, dm.$$

Proof

We'll come back

(2) $\implies$ (3) This is proven in Theorem 74. $\square$

30.2 *The L^p-spaces*

It is helpful to review some of the contents from real analysis on metric spaces, particularly in

- metric;
- convergence;
- Cauchy sequences; and
- completeness.

You may find the material at PMATH 351.

It may also be helpful to review some of the materials on L^p spaces in the course on Lebesgue measure and integration (PMATH 450).

Definition 46 ($\mathcal{L}^p$-spaces)

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Let $1 \leq p < \infty$. We define

$$\mathcal{L}^p(X, \mathfrak{M}, \mu) := \left\{ f : X \to \mathbb{R}_e \text{ measurable } : \int_X |f|^p \, d\mu < \infty \right\}.$$

We set

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}}.$$

Definition 47 (Essentially Bounded)

Let $f : X \to \mathbb{R}_e$ be a measurable function. We say that f is ***essentially bounded*** *if*

$$\exists M \quad \mu(\{x : |f(x)| > M\}) = 0.$$

Definition 48 (Essential Norm)

Let $f : X \to \mathbb{R}_e$ be a measurable function. We define the ***essential norm*** *of f as*

$$\|f\|_\infty := \inf\{M : \mu(\{x : |f(x)| > M\}) = 0\}.$$

Note 30.2.1

While we call $\|f\|_\infty$ the essential ***norm****, we have not defined what a norm is. We shall do that later.*

Definition 49 ($\mathcal{L}^\infty$-space)

We define the $\mathcal{L}^\infty$-space as

$$\mathcal{L}^\infty(X, \mathfrak{M}, \mu) := \{ f : X \to \mathbb{R}_e \text{ measurable } : f \text{ is essentially bounded} \}.$$

Definition 50 (Hölder conjugates)

Let $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. We say that (p, q) is a ***Hölder conjugate*** *if*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

where we set

$$p = 1,\ q = \infty \quad \text{and} \quad p = \infty,\ q = 1.$$

Lemma 76 (Young's Inequality)

Let (p, q) be a Hölder conjugate, and $a, b \geq 0$. Then

$$a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof

We'll come back □

Theorem 77 (Hölder's Inequality)

Let (p, q) *be a Hölder conjugate. If* $f \in \mathcal{L}^p(X, \mathfrak{M}, \mu)$ *and* $g \in \mathcal{L}^q(X, \mathfrak{M}, \mu)$*, then* $f \cdot g \in \mathcal{L}^1(X, \mathfrak{M}, \mu)$ *and*

$$\left\| f \cdot g \right\|_1 \leq \left\| f \right\|_p \cdot \left\| g \right\|_q .$$

Proof

We'll come back $\square$

31 Lecture 31 Nov 22th 2019

31.1 The L^p-spaces (Continued)

Proposition 78 ($\mathcal{L}^p$-spaces are Vector Spaces)

Let $1 \leq p \leq \infty$. $\mathcal{L}^p(X, \mathfrak{M}, \mu)$ is a vector space.

Proof

Case: $1 < p < \infty$ Let $\alpha \in \mathbb{R}$ and $f, g \in \mathcal{L}^p$.

Closure under scalar multiplication Notice that

$$\int_X |\alpha f|^p \, d\mu = |\alpha|^p \int_X |f|^p \, d\mu < \infty.$$

Thus $\alpha f \in \mathcal{L}^p$.

Closure under addition First, notice that

$$\left| f(x) + g(x) \right| \leq \begin{cases} 2\left| f(x) \right| & \left| f(x) \right| \geq \left| g(x) \right| \\ 2\left| g(x) \right| & \text{otherwise} \end{cases}.$$

Thus

$$\begin{aligned} \left| f(x) + g(x) \right|^p &\leq \begin{cases} 2^p \left| f(x) \right|^p & \left| f(x) \right| \geq \left| g(x) \right| \\ 2^p \left| g(x) \right|^p & \text{otherwise} \end{cases} \\ &\leq 2^p \left(\left| f(x) \right|^p + \left| g(x) \right|^p \right). \end{aligned}$$

Hence

$$\int_X |f+g|^p \, d\mu \leq \int_X 2^p \left(|f(x)|^p + |g(x)|^p \right) d\mu < \infty.$$

It follows that $f + g \in \mathcal{L}^p$. □

Theorem 79 (Minkowski's Inequality)

Let $1 \leq p \leq \infty$. If $f, g \in \mathcal{L}^p$, then

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p .$$

Proof

Case: $1 < p < \infty$ Let (p, q) be a Hölder conjugate. Note that $q = \frac{p}{p-1}$ and $lp = (p-1)q$. Then, using Hölder's Inequality at $(*)$, we have

$$\begin{aligned}
\|f+g\|_p^p &= \int_X |f(x)+g(x)|^p \, d\mu \\
&= \int_X |f(x)+g(x)| \, |f(x)+gx|^{p-1} \, d\mu \\
&\leq \int_X |f(x)| \, |f(x)+g(x)|^{p-1} \, d\mu + \int_X |g(x)| \, |f(x)+g(x)|^{p-1} \, d\mu \\
&\overset{(*)}{\leq} \left(\int_X |f(x)|^p \, d\mu \right)^{\frac{1}{p}} \left(\int_X |f(x)+g(x)|^{(p-1)q} \, d\mu \right)^{\frac{1}{q}} \\
&\qquad + \left(\int_X |g(x)|^p \, d\mu \right)^{\frac{1}{p}} \left(\int_X |f(x)+g(x)|^{(p-1)q} \, d\mu \right)^{\frac{1}{q}} \\
&= \|f\|_p \left(\int_X |f(x)+g(x)|^p \right)^{\frac{1}{q}} + \|g\|_p \left(\int_X |f(x)+g(x)|^p \right)^{\frac{1}{q}} \\
&= (\|f\|_p + \|g\|_p) \left(\int_X |f(x)+g(x)|^p \right)^{\frac{1}{q}} \\
&= (\|f\|_p + \|g\|_p) \|f+g\|^{\frac{p}{q}} .
\end{aligned}$$

Hence

$$\|f+g\|_p^{p-\frac{p}{q}} \leq \|f\|_p + \|g\|_p .$$

Finally, note that

$$p - \frac{p}{q} = p\left(1 - \frac{1}{q}\right) = p \cdot \frac{1}{p} = 1.$$

Thus

$$\left\| f + g \right\|_p \leq \left\| f \right\|_p + \left\| g \right\|_p . \qquad \square$$

Definition 51 (Semi-norm)

Let $\mathcal{V}$ be a vector space over $\mathbb{C}$. We call the function $\|\cdot\| : \mathcal{V} \to \mathbb{R}$ a ***semi-norm*** *when $\|\cdot\|$ satisfies*

1. $\forall v \in \mathcal{V} \quad \|v\| \geq 0$;
2. $\forall v \in \mathcal{V} \;\; \forall \alpha \in \mathbb{C} \quad \|\alpha v\| = |\alpha| \, \|v\|$; *and*
3. *(**triangle inequality**)* $\forall v, w \in \mathcal{V} \quad \|v + w\| \leq \|v\| + \|w\|$.

Definition 52 (Norm)

Let $\mathcal{V}$ be a vector space over $\mathbb{C}$. We call the function $\|\cdot\| : \mathcal{V} \to \mathbb{R}$ a ***norm*** *when $\|\cdot\|$ is a semi-norm such that*

$$\|v\| = 0 \iff v = 0.$$

Definition 53 (Normed Space)

Let $\mathcal{V}$ be a vector space over $\mathbb{C}$ and $\|\cdot\|$ be a norm on $\mathcal{V}$. We call the pair $(\mathcal{V}, \|\cdot\|)$ a ***normed space****.*

Note 31.1.1

Let $\|\cdot\|$ be a semi-norm for a vector space $\mathcal{V}$. Then the set

$$\mathcal{N} := \{v \in \mathcal{V} : \|v\| = 0\}$$

is a subspace. We may then consider the quotient space

$$\mathcal{V} / \mathcal{N} := \{v + \mathcal{N} : v \in \mathcal{V}\} = \{[v] : v \sim w\},$$

where $v \sim w \iff v - w \in \mathcal{N}$. [1] *We can show that $\mathcal{V} / \mathcal{N}$ is a vector space. We may then set*

$$\|[v]\| := \|v\|,$$

which will then be a ***norm*** *on $\mathcal{V} / \mathcal{N}$.*

Most importantly to us, notice that given $f \in \mathcal{L}^p$, we know that

$$\|f\|_p = 0 \iff \int_X |f|^p \, d\mu = 0 \iff f = 0 \text{ a.e.}$$

[1] We shall not show that $\mathcal{V} / \mathcal{N}$ is a vector space, but this is not hard, and those that are familiar with vector spaces will quickly realize so.

Definition 54 (L^p-spaces)

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Let $1 \leq p \leq \infty$. Consider the space $\mathcal{L}^p(X, \mathfrak{M}, \mu)$. Let

$$\mathcal{N}^p := \{f \in \mathcal{L}^p : f = 0 \text{ a.e.}\}.$$

We define

$$\begin{aligned} L^p(X, \mathfrak{M}, \mu) &:= \mathcal{L}^p(X, \mathfrak{M}, \mu) \Big/ \mathcal{N}^p(X, \mathfrak{M}, \mu) \\ &= \{[f] : f \sim g \iff f = g \text{ a.e.}\}. \end{aligned}$$

Note 31.1.2

We define the ***norm*** *of L^p-spaces as we do in the last note, i.e. we define*

$$\|[f]\|_p := \|f\|_p$$

for any $[f] \in L^p(X, \mathfrak{M}, \mu)$. For $1 \leq p < \infty$, Minkowski's Inequality tells us that $(L^p(X, \mathfrak{M}, \mu), \|\cdot\|_p)$ is a normed space. We leave the case of $p = \infty$ to be a homework problem for a later date.

Remark 31.1.1 (Normed spaces are metric spaces)

It is important that we note the following. [2] *A norm* ***induces*** *a metric, and so normed spaces are also metric spaces. In particular, given a normed vector space* $(\mathcal{V}, \|\cdot\|)$*, the standard metric of the vector space is given by*

$$\rho(v, w) = \|v - w\|,$$

and we can show that $(\mathcal{V}, \rho)$ *is a metric space.*

[2] This requires knowledge from metric spaces. See PMATH 351.

Definition 55 (Banach Space)

A normed vector space $(\mathcal{V}, \|\cdot\|)$ *is called a* ***Banach space*** *if* $(\mathcal{V}, \rho)$ *is complete, where* ρ *is the induced metric*

$$\rho(v, w) = \|v - w\|.$$

Note that completeness is in the context of metric spaces, where every Cauchy sequence converges to a point in the space.

Theorem 80 (Riesz-Fischer Theorem)

Let $(X, \mathfrak{M}, \mu)$ *be a measure space. For any* $1 \leq p \leq \infty$, $(L^p(X, \mathfrak{M}, \mu), \|\cdot\|_p)$ *is a Banach space.*

Proof

Case: $1 \leq p < \infty$ Let $\{[f_n]\}_n \subseteq L^p$ be a Cauchy sequence. We need to construct an $f \in \mathcal{L}^p$ such that $\left\|[f] - [f_n]\right\|_p \to 0$ as $n \to \infty$.

By the Cauchyness of the arbitrary sequence, inductively define a subsequence $\{N_1 < N_2 < \ldots\}$ such that $\forall n, m \geq N_k$ we have

$$\left\|[f_n] - [f_m]\right\|_p < \frac{1}{2^k}.$$

Set

$$g_1 = f_{N_1},\ g_2 = f_{N_2} - f_{N_1},\ \ldots,\ g_k = f_{N_k} - f_{N_{k-1}},\ \ldots.$$

Notice that

$$\left\|g_k\right\|_p = \left\|[g_k]\right\|_p = \left\|[f_{N_k}] - [f_{N_{k-1}}]\right\|_p < \frac{1}{2^{k-1}}$$

It thus follows that

$$\sum_{k=1}^{\infty} \left\|g_k\right\|_p < \infty.$$

Let $C = \sum_{k=1}^{\infty} \left\|\left|g_k\right|\right\|_p < \infty$. Let $h_n(x) = \sum_{k=1}^{n} \left|g_k(x)\right|$. It is clear that

$$h_1(x) \leq h_2(x) \leq h_3(x) \leq \ldots.$$

Set $h(x) = \sum_{k=1}^{\infty} \left|g_k(x)\right| = \lim_{n\to\infty} h_n(x)$ (which possibly evaluates to ∞). Then for any m,

$$\|h_m(x)\|_p = \left\|\sum_{k=1}^{m} \left|g_k(x)\right|\right\|_p \leq \sum_{k=1}^{m} \left\|\left|g_k(x)\right|\right\|_p \leq C.$$

In other words, $\forall m$, we have

$$\int_X h_m(x)^p \, d\mu = \|h_m(x)\|_p \leq C^p.$$

[3] Furthermore, we know that $h_m(x)^p \nearrow h(x)^p$. By the ★ Monotone Convergence Theorem (MCT), we have

[3] Note that we may drop the absolute value on $h_m(x)$ since $h_m(x) \geq 0$.

$$\int_X h(x)^p \, d\mu = \lim_{m\to\infty} \int_X h_m(x)^p \, d\mu \leq C^p.$$

This means that $\mu(\{x : h(x) = \infty\}) = 0$.

Let $Y = X \setminus \{x : h(x) = \infty\}$. Then $\forall x \in Y$, we have $h(x) < \infty$. This means that $\forall x \in Y$,

$$h(x) = \sum_{k=1}^{\infty} \left|g_k(x)\right| < \infty,$$

i.e. $\{g_k\}_k$ is absolutely convergent on Y. Hence $\{g_k\}_k$ converges on Y. With that, we define

$$f(x) := \begin{cases} \sum_{k=1}^{\infty} g_k(x) & x \in Y \\ 0 & x \notin Y \end{cases}$$

Then $\forall x \in Y$, we have

$$f(x) = \lim_{K \to \infty} \sum_{k=1}^{K} g_k(x) = \lim_{K \to \infty} f_{N_k}(x).$$

Finally,

$$\begin{aligned}
\left\| f - \sum_{k=1}^{K} g_k \right\|_p &= \left(\int_X \left| f(x) - \sum_{k=1}^{K} g_k(x) \right|^p \, d\mu \right)^{\frac{1}{p}} \\
&= \left(\int_Y \left| f(x) - \sum_{k=1}^{K} g_k(x) \right|^p \, d\mu \right)^{\frac{1}{p}} \\
&= \left(\int_Y \left| f(x) - f_{N_k}(x) \right|^p \, d\mu \right)^{\frac{1}{p}} \\
&= \left(\int_Y \left| \sum_{k=K+1}^{\infty} g_k(x) \right|^p \, d\mu \right)^{\frac{1}{p}} \\
&= \left\| \sum_{k=K+1}^{\infty} g_k \right\|_p \leq \sum_{k=K+1}^{\infty} \left\| g_k \right\|_p,
\end{aligned}$$

where we observe that the final term is the tail of C, and hence goes to 0 as $K \to \infty$. Hence $\left\| f - f_{N_k} \right\|_p \to \infty$, and so

$$\left\| [f] - [f_n] \right\|_p \to 0 \quad \text{as} \quad n \to \infty. \qquad \square$$

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32.1 The L^p-spaces (Continued 2)

Homework (Homework 24)

Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. Prove that $\exists F_i : \mathbb{R} \to \mathbb{R}$ increasing and right continuous for $i = 1, 2$ so that $F = F_1 + F_2$, $\mu_{F_1} \ll m$ and $\mu_{F_2} \perp m$.

Homework (Homework 25)

Let $f \in \mathcal{L}^\infty(X, \mathfrak{M}, \mu)$. Prove that

$$\mu(\{x : |f(x)| > \|f\|_\infty\}) = 0$$

and that if $\|f\|_\infty > 0$, then

$$\|f\|_\infty = \sup\{M : \mu(\{x : |f(x)| > M\}) > 0\}.$$

Homework (Homework 26)

Let $f, g \in \mathcal{L}^\infty(X, \mathfrak{M}, \mu)$. Prove that $f + g, f \cdot g \in \mathcal{L}^\infty(X, \mathfrak{M}, \mu)$ and

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$
$$\|f \cdot g\|_\infty \leq \|f\|_\infty \cdot \|g\|_\infty .$$

Homework (Homework 27)

Prove $(L^\infty(X, \mathfrak{M}, \mu), \|\cdot\|_\infty)$ is a Banach space.

Homework (Homework 28)

Let $0 < \alpha$, and define $f_\alpha : [0,1] \to \mathbb{R}$ by $f_\alpha(x) = x^\alpha \sin\left(\frac{1}{x}\right)$. Find and prove the values of α such that

1. $f_\alpha \in \mathrm{BV}[0,1]$.
2. $f_\alpha \in \mathrm{AC}[0,1]$.

Theorem 81 (Density of Simple Functions in $\mathcal{L}^p$)

Let $1 \leq p \leq \infty$, $f \in \mathcal{L}^p(X, \mathfrak{M}, \mu)$ and $\varepsilon > 0$. Then $\exists \psi \in \mathcal{L}^p(X, \mathfrak{M}, \mu)$ a simple function such that $\|f - \psi\|_p < \varepsilon$.

Proof

Case: $1 < p < \infty$ Write $f = f^+ - f^-$. By Theorem 25, there exists simple functions

$$\varphi_n \nearrow f^+ \quad \text{and} \quad \psi_n \nearrow f^-,$$

where $\varphi_n, \psi_n \in \mathcal{L}^p(X, \mathfrak{M}, \mu)$ since $\varphi_n \leq f \implies \int_X \varphi_n \, d\mu \leq \int_X f^+ \, d\mu < \infty$ and similarly so for ψ_n with f^-.

Let $\gamma_n = \varphi_n - \psi_n$. Then

$$\begin{aligned} |f - \gamma_n|^p &= |f^+ - \varphi_n - (f^- - \psi_n)|^p \\ &= |f^+ - \varphi_n|^p + |f^- - \psi_n|^p \\ &\leq |f^+|^p + |f^-|^p = \|f\|_p . \end{aligned}$$

because of how the pairs alternate

By the Monotone Convergence Theorem (MCT),

$$\lim_{n\to\infty} \int_X \left|f - \gamma_n\right|^p \, d\mu = \int_X \lim_{n\to\infty} \left|f - \gamma_n\right|^p \, d\mu = 0.$$

Thus $\left\|f - \gamma_n\right\|_p^p \to 0$, i.e. $\exists n_0$ such that $\left\|f - \gamma_n\right\|_p < \varepsilon$. □

Remark 32.1.1

This density extends to L^p-spaces.

Definition 56 (Bounded Linear Functional)

Let $(\mathcal{V}, \|\cdot\|)$ be a normed vector space. A linear function $T : \mathcal{V} \to \mathbb{R}$ is called a ***linear functional****. It is called a* ***bounded linear functional*** *if $\exists M$ such that $|T(v)| \leq M \cdot \|v\|$ for all $v \in \mathcal{V}$.*

We define

$$\|T\| := \inf\{M : |T(v)| \leq M \cdot \|v\|, \, v \in \mathcal{V}\}.$$

Alternatively,

$$\begin{aligned} \|T\| &:= \sup\{|T(v)| : \|v\| = 1\} \\ &= \sup\{|T(v)| : \|v\| \leq 1\} \\ &= \sup\left\{\frac{|T(v)|}{\|v\|} : v \neq 0\right\}. \end{aligned}$$

Proposition 82 (Bounded Linear Functionals are Continuous)

Let $T : \mathcal{V} \to \mathbb{R}$ be a bounded linear functional. Then T is continuous. [1]

[1] The converse is also true, and we have seen this in PMATH 351 and PMATH 450.

Proof

Given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{\|T\|}$. Then if $d(v, w) = \|v - w\| < \delta$, we have

$$|T(v) - T(w)| = \|T(v - w)\| \leq \|T\| \cdot \|v - w\| < \|T\| \cdot \frac{\varepsilon}{\|T\|} = \varepsilon.$$ □

Proposition 83 (p-norms As Bounded Linear Functionals)

Let (p, q) be a Hölder conjugate, and $g \in \mathcal{L}^q(X, \mathfrak{M}, \mu)$. Then the function $T_g : L^p(X, \mathfrak{M}, \mu) \to \mathbb{R}$ defined by $T_g[f] := \int_X f g \, d\mu$ is a bounded linear functional with $\left\| T_g \right\| = \left\| g \right\|_q$. Moreover,

$$T_{g_1} = T_{g_2} \iff g_1 = g_2 \text{ a.e.}$$

Proof

Case: $1 < p < \infty$ That T_g is linear is by linearity of integration. Furthermore, note that if $f_1 = f_2$ a.e., then $f_1 g = f_2 g$ a.e. and so

$$T_g[f_1] = \int_X f_1 g \, d\mu = \int_X f_2 g \, d\mu = T_g[f_2].$$

Thus T_g is well-defined.

By Hölder's Inequality, $\forall f \in \mathcal{L}^p(X, \mathfrak{M}, \mu)$, we have

$$\left| T_g[f] \right| = \left| \int_X f g \, d\mu \right| \leq \left\| f \right\|_p \cdot \left\| g \right\|_q = \left\| [f] \right\|_p \cdot \left\| [g] \right\|_q .$$

By definition (the last of the alternatives), we know that T_g is bounded and $\left\| T_g \right\| \leq \left\| g \right\|_q$.

[2] Consider $f(x) = \operatorname{sgn}(g(x)) \cdot \left| g(x) \right|^{q-1}$. Then $f(x) g(x) = \left| g(x) \right|^q$. This means that

$$\left\| g \right\|_q^q = \left(\int_X f g \, d\mu \right) = \left| T_g[f] \right| \leq \left\| f \right\|_p \left\| g \right\|_q .$$

[2] We consider one example where the other inequality is true, and this is sufficient to show that the other inequality is true.

Note that

$$\left\| f \right\|_p^p = \int_X |f|^p \, d\mu = \int_X |g|^{p(q-1)} \, d\mu = \int_X |g|^q \, d\mu = \left\| g \right\|_q^q .$$

Hence

$$\left\| T_g \right\| \geq \frac{\left| T_g[f] \right|}{\left\| f \right\|_p} = \frac{\left\| g \right\|_q^q}{\left\| g \right\|_q^{q/p}} = \left\| g \right\|_q^{q - q/p} = \left\| g \right\|_q .$$

Hence we must have $\left\| T_g \right\| = \left\| g \right\|_q$. □

Theorem 84 (Riesz Representation Theorem)

Let $1 \leq p < \infty$ and $(X, \mathfrak{M}, \mu)$ be a σ-finite measure space. If $T : L^p(X, \mathfrak{M}, \mu) \to \mathbb{R}$ is a bounded linear functional, then $\exists g \in \mathcal{L}^q$ such that $T = T_g$ and $\left\|g\right\|_q = \|T\|$.

Proof

Case: $1 < p < \infty$

Case: $\mu(X) < \infty$ Since μ is finite, $\forall E \in \mathfrak{M}$ such that

$$\|\chi_E\|_p^p = \int_X \chi_E^p \, d\mu = \int_X \chi_E \, d\mu = \mu(E).$$

In other words, all characteristic functions of subsets of X are in the $\mathcal{L}^p$-space.

Let $\nu : \mathfrak{M} \to \mathbb{R}$ by $\nu(E) = T(\chi_E)$.

Claim: ν is a finite signed measure To be proven

Constructing $g \in \mathcal{L}^q$ Note that $\forall E \in \mathfrak{M}$, $\mu(E) = 0 \implies \|\chi_E\|_p = 0 \implies \chi_E = 0$ a.e. This means that

$$\mu(E) = T([\chi_E]) = T([0]) = 0.$$

Thus $\nu \ll \mu$.

Taking a Hahn Decomposition, write $X = A \cup B$, where A is a positive set and B a negative set. Let $\nu^+(E) = \nu(E \cap A)$ and $\nu^-(E) = -\nu(E \cap B)$. By Lemma 59, we know that $\nu^+, \nu^- \ll \mu$. By the Radon-Nikodym theorem, $\exists g^+, g^- : X \to [0, \infty]$ such that

$$\nu^+(E) = \int_E g^+ \, d\mu \quad \text{and} \quad \nu^-(E) = \int_E g^- \, d\mu.$$

Then

$$\nu(E) = \int_E (g^+ - g^-) \, d\mu = \int_X \chi_E (g^+ - g^-) \, d\mu = T([\chi_E]).$$

It follows that for any simple function ψ,

$$T([\psi]) = \int_X \psi(g^+ - g^-) \, d\mu. \tag{32.1}$$

Let $g = g^+ - g^-$. Note that $g^+ \restriction_B = 0$ and $g^- \restriction_A = 0$.

Since $g^+, g^- \geq 0$, by Theorem 25, $\exists \varphi_n \nearrow g^+$ and $\psi_n \nearrow g^-$. Let $\gamma_n = \varphi_n^{q/p} - \psi_n^{q/p}$. Then $\gamma_n(x) \to g(x)^{q/p}$ as $n \to \infty$. Observe that for each n, we have

$$\begin{aligned} T([\gamma_n]) &= \int_X \gamma_n g \, d\mu = \int_X \left(\varphi_n^{q/p} - \psi_n^{q/p} \right) (g^+ - g^-) \, d\mu \\ &= \int_X \varphi_n^{q/p} g^+ \, d\mu + \int_X \psi_n^{q/p} g \, d\mu, \end{aligned}$$

since $\varphi_n^{q/p} g^- = 0 = \psi_n^{q/p} g^+$ for all x. Note that for (p, q) a Hölder conjugate, we have $\frac{q}{p} + 1 = q$. Thus, with the Monotone Convergence Theorem (MCT),

$$\begin{aligned} \lim_n T([\gamma_n]) &= \lim_n \int_X \varphi_n^{q/p} g^+ + \psi_n^{q/p} g^- \, d\mu \\ &= \int_X (g^+)^{q/p+1} + (g^-)^{q/p+1} \, d\mu \\ &= \int_X |g|^q \, d\mu = \|g\|_q^q. \end{aligned}$$

Note that this means we have

$$\|T\| \leq \|g\|_q^q.$$

Observe that

$$\|g\|_q^q = \lim_n T([\gamma_n]) \leq \|T\| \cdot \lim_n \|\gamma_n\|_p. \tag{32.2}$$

Note that

$$\lim_n \int_X |\gamma_n|^p \, d\mu = \lim_n \int_X \left| \varphi_n^q + \psi_n^q \right| d\mu \overset{MCT}{=} \int_X |g|^q \, d\mu.$$

Thus

$$\lim_n \|\gamma_n\|_p = \left(\int_X |g|^q \, d\mu \right)^{\frac{1}{p}} = \|g\|_q^{q/p}.$$

Going back to Equation (32.2), we have

$$\left\|g\right\|_q^q \leq \|T\| \cdot \left\|g\right\|_q^{q/p},$$

which implies that

$$\left\|g\right\|_q = \left\|g\right\|_q^{q-q/p} \leq \|T\|.$$

We have thus constructed g such that $g \in \mathcal{L}^q$.

$T = T_g$ Recall that we defined

$$T_g([f]) \coloneqq \int_X f g \, d\mu,$$

which is a bounded linear functional on L^p. By Equation (32.1), we showed that $T_g([\psi]) = T([\psi])$ for any simple function ψ. By the density of simple functions in L^p, and bounded linear functionals being continuous, we have that $\forall [f] \in L^p$, $T_g([f]) = T([f])$. Furthermore, note that there is uniqueness for g, for if

$$\nu(E) = \int_E g_1 \, d\mu = \int_E g_2 \, d\mu,$$

then $g_1 = g_2$ a.e.

Note that this also shows that $\left\|g\right\|_q = \|T\|$ vis-á-vis Proposition 83.

This completes the case for $\mu(X) < \infty$. ⊣

Case: μ is σ-finite (sketch proof) We may write $X = \dot{\bigcup}_n X_n$ where $\mu(X_n) < \infty$ for all n. The key idea is to extend each $f : X_n \to \mathbb{R}$ by defining $\tilde{f} : X \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f(x) & x \in X_n \\ 0 & x \notin X_n \end{cases}.$$

Then

$$\int_{X_n} \left|f\right|^p \, d\mu = \int_X \left|\tilde{f}\right|^p \, d\mu.$$

Using the one-to-one identification, from $L^p(X_n, \mathfrak{M}, \mu) \to L^p(X, \mathfrak{M}, \mu)$

given by $f \to \tilde{f}$, we note that

$$T \restriction_{L^p(X_n, \mathfrak{M}, \mu)} : L^p(X_n, \mathfrak{M}, \mu) \to \mathbb{R}$$

remains a bounded linear functional. Then by the last case for when $\mu(X_n) < \infty$, for each n, $\exists ! [g_n] \in L^q(X_n, \mathfrak{M}, \mu)$ with $g_n : X_n \to \mathbb{R}$ such that

$$\left\| g_n \right\|_q^q = \int_{X_n} \left| g_n \right|^q \, d\mu < \infty \quad \text{and} \quad T([\tilde{f}]) = \int_{X_n} f g_n \, d\mu.$$

We can then stitch each g_n's together to get $g : X \to \mathbb{R}$ given by

$$g(x) = g_n(x) \text{ for when } x \in X_n.$$

From here, it remains to show that $g \in \mathcal{L}^q(X, \mathfrak{M}, \mu)$ and that $T_g([f]) = T([f])$ for all $[f] \in L^p$. □

Remark 32.1.2

1. *Note that Riesz Representation Theorem is false if X is not σ-finite.*
2. *Even when X is σ-finite, the theorem is false for when $p = \infty$. In particular, $\exists T : L^\infty(X, \mathfrak{M}, \mu) \to \mathbb{R}$ a bounded linear functional such that $\nexists g \in \mathcal{L}^1$ such that $T([f]) = \int_X f g \, d\mu$. This is also the case for when $X = \mathbb{N}$, and μ is the counting measure.*

A Deep Dives into Proofs

A.1 Proving that $\mathfrak{M}$ is closed under countable unions in Carathéodory's Theorem

This section is created in reference to the proof for Carathéodory's Theorem.

We have

$$\mathfrak{M} = \{A \subseteq X : A \text{ is } \mu^*\text{-measurable }\}$$

where μ^* is an outer measure. We wanted to show that $\mathfrak{M}$ is a σ-algebra. In particular, the hard problem was to show that $\mathfrak{M}$ is closed under countable unions.

Consider $\{A_n\}_n \subseteq \mathfrak{M}$. Thinking from behind, WTS $\forall E \subseteq X$,

$$\begin{aligned} \mu^*(E) &\geq \mu^*\left(E \cap \bigcup_n A_n\right) + \mu^*\left(E \cap \left(\bigcup_n A_n\right)^C\right) \\ &= \mu^*\left(E \cap \bigcup_n A_n\right) + \mu^*\left(E \cap \left(\bigcap_n A_n^C\right)\right). \end{aligned}$$

For simplicity, write $B = \bigcup_n A_n$. WTS

$$\mu^*(E) \geq \mu^*(E \cap B) + \mu^*(E \cap B^C). \tag{$*$}$$

Also thinking from behind, if $\mathfrak{M}$ is a σ-algebra, [1] then it must be an algebra (of sets). We showed that $\mathfrak{M}$ is closed under complementation.

[1] Useful links: Algebra of Sets, σ-Algebra of Sets.

If $\mathfrak{M}$ is closed under finite unions, [2] then for each $N \in \mathbb{N}$,

[2] Unproved point 1

$$\mu^*(E) = \mu^*\left(E \cap \bigcup_{n=1}^{N} A_n\right) + \mu^*\left(E \cap \left(\bigcup_{n=1}^{N} A_n\right)^C\right).$$

Let $B_N \coloneqq \bigcup_{n=1}^{N} A_n \in \mathfrak{M}$. Then

$$\mu^*(E) = \mu^*(E \cap B_N) + \mu^*(E \cap B_N^C) \tag{†}$$

for each $N \in \mathbb{N}$.

Notice that

$$B_N = \bigcup_{n=1}^{N} A_n \subseteq \bigcup_{n=1}^{\infty} A_n = B.$$

Consequently,

$$\implies B^C \supseteq B_N^C \implies \mu^*(B^C) \le \mu^*(B_N^C)$$

by the monotonicity of the outer measure.

As a result, looking at Equation (∗) and Equation (†), we see that

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap B_N) + \mu^*(E \cap B_N^C) \\ &\geq \mu^*(E \cap B_N) + \mu^*(E \cap B^C) \end{aligned}$$

for each $N \in \mathbb{N}$.

We are in quite the predicament at this point. We need to do something about $\mu^*(E \cap B_N)$ and somehow relate it to $\mu^*(E \cap B)$. We can try and see that

$$\mu^*(E \cap B_N) \le \sum_{n=1}^{N} \mu^*(E \cap A_n).$$

Notice that in the case of **equality**, we would have

$$\mu^*(E) \geq \sum_{n=1}^{N} \mu^*(E \cap A_n) + \mu^*(E \cap B^C)$$

for all $N \in \mathbb{N}$. Since $\{\sum_{n=1}^{N} \mu^*(E \cap A_n)\}_N$ is an increasing sequence in $\mathbb{R}$, we have

$$\mu^*(E) \geq \sum_{n=1}^{\infty} \mu^*(E \cap A_n) + \mu^*(E \cap B^C),$$

and

$$\sum_{n=1}^{\infty} \mu^*(E \cap A_n) \geq \mu^*(E \cap B)$$

by subadditivity since $B = \bigcup_{n=1}^{\infty} A_n$.

Unfortunately, the equality does not always hold. But, since μ^* is an outer measure, we can make an educated guess [3] that given $\{A_n\}_n$ a disjoint collection of sets,

[3] Unproved point 2

$$\mu^*\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \mu^*(A_n).$$

Our work becomes even easier with the realization of Homework 4. Proving that all of our above argument works for the case of $\{A_n\}_n \subseteq \mathfrak{M}$ being disjoint, is sufficient to prove that $\mathfrak{M}$ is indeed a σ-algebra.

B Common Themes and Tricks

B.1 Re-represent an arbitrary union using disjoint sets

A common trick in measure theory, especially when it comes to a collection of sets, is to represent its union as a disjoint union of sets. This is a useful trick because measures simply add over disjoint sets, instead of just having **subadditivity**.

Example B.1.1

Given a collection $\{A_n\}_n$ of sets, we may define a collection of disjoint sets whose union is $\bigcup_n A_n$ as such:

$$\begin{aligned}
F_1 &= A_1 \\
F_2 &= A_2 \setminus A_1 \\
F_3 &= A_3 \setminus (A_1 \cup A_2) \\
&\vdots \\
F_n &= A_n \setminus \bigcup_{i=1}^{n-1} A_i \\
&\vdots
\end{aligned}$$

Example B.1.2

Given an increasing collection $\{A_n\}_n$ of sets, i.e.

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots,$$

we may represent the countable union of the A_n's as such: let

$$\begin{aligned} F_1 &= A_1 \\ F_2 &= A_2 \setminus A_1 \\ F_3 &= A_3 \setminus A_2 \\ &\vdots \\ Fn &= A_n \setminus A_{n-1} \\ &\vdots \end{aligned}$$

Remark B.1.1

The reason why we simply consider $F_n = A_n \setminus A_{n-1}$ instead of having to take a union up to the $(n-1)^{th}$ set in Example B.1.2 is because

$$\bigcup_{i=1}^{n-1} A_i = A_{n-1}. \tag{B.1}$$

The reader may also notice that Example B.1.2 is an application of Example B.1.1 just because of Equation (B.1).

B.2 Abusing σ-algebras

When we discuss about a property within the realm of σ-algebras, one should remain aware that one of the options available to them when working on a proof, is to show that the set that contains elements that allows the property to hold is itself a σ-algebra.

In particular, if P is the property of which we want to show is true, then we may be able to show that

$$\mathcal{A} := \{x : P(x)\}$$

is a σ-algebra to complete our proof.

Bibliography

Folland, G. B. (1999). *Real Analysis: Modern Techniques and Their Applications*. John Wiley and Sons, Inc., 2nd edition.

Halmos, P. R. (1974). *Graduate Text in Mathematics: Measure Theory*. Springer-Verlag.

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