PMATH450 — Lebesgue Integration and Fourier Analysis

Classnotes for Spring 2019

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Preface

The pre-requisite to this course is Real Analysis. We will use a lot of the concepts introduced in Real Analysis, at times without explicitly stating it. Refer to notes on PMATH351.

This course is spiritually broken into 2 pieces:

- Lebesgue Integration; and
- Fourier Analysis,

which is as the name of the course.

In this set of notes, we use a special topic environment called **culture** to discuss interesting contents related to the course, but will not be throughly studied and not tested on exams.

For some unknown reason, mysterious glyphs are replacing common math characters in an inconsistent way, and I have not the faintest idea how this is happening, or why this is happening. The dark version of the notes does not seem to have this problem. Please use that version of the notes for a cleaner reference.

If you have any idea what is causing the weird glitch, or a solution, please shoot me an issue at `https://gitlab.com/japorized/TeX_notes/issues`.

1 Lecture 1 May 07th 2019

Since many of our results work for both $\mathbb{C}$ and $\mathbb{R}$, we shall use $\mathbb{K}$ throughout this course to represent either $\mathbb{C}$ or $\mathbb{R}$.

1.1 Riemannian Integration

Definition 1 (Norm and Semi-Norm)

*Let V be a vector space over $\mathbb{K}$. We define a **semi-norm** on V as a function*

$$\nu : V \to \mathbb{R}$$

that satisfies

1. *(**Positive Semi-Definite**) $v(x) \geq 0$ for all $x \in V$;*
2. *$\nu(\kappa x) = |\kappa| \, \nu(x)$ for any $\kappa \in \mathbb{K}$ and $x \in V$; and*
3. *(**Triangle Inequality**) $\nu(x + y) \leq \nu(x) + \nu(y)$ for all $x, y \in V$.*

*If $\nu(x) = 0 \implies x = 0$, then we say that ν is a **norm**. In this case, we usually write $\|\cdot\|$ to denote the norm, instead of ν.*

Note 1.1.1

- *We sometimes call a semi-norm a **pseudo-length**.*

Remark 1.1.1

Notice that we wrote $\nu(x) = 0 \implies x = 0$ instead of $\nu(x) = 0 \iff x = 0$. This is because if $z = 0 \in V$, then

$$v(z) = v(0z) = 0.$$

Exercise 1.1.1

Show that if ν is a semi-norm on a vector space V, then $\forall x, y \in V$,

$$|\nu(x) - \nu(y)| \leq \nu(x - y).$$

Proof

Notice that by condition (2) and (3), we have

$$\nu(x - y) \leq \nu(x) + \nu(-y) = \nu(x) - \nu(y),$$

and

$$\nu(x - y) = -\nu(y - x) \geq -(\nu(y) - \nu(x)) = \nu(x) - \nu(y).$$

It follows that indeed

$$|\nu(x) - \nu(y)| \leq \nu(x - y).$$

□

Example 1.1.1

The absolute value $|\cdot|$ is a **norm** on $\mathbb{K}$.

Example 1.1.2 (p-norms)

Consider $N \geq 1$ an integer. We define a family of norms on

$$\mathbb{K}^N = \underbrace{K \times K \times \ldots \times K}_{N \text{ times}}.$$

1-norm

$$\left\| (x_n)_{n=1}^N \right\|_1 := \sum_{n=1}^N |x_n|.$$

Infinity-norm, ∞-norm

$$\left\| (x_n)_{n=1}^N \right\|_\infty := \max_{1 \le n \le N} |x_n| .$$

Euclidean-norm, 2-norm

$$\left\| (x_n)_{n=1}^N \right\|_2 := \left(\sum_{n=1}^N |x_n|^2 \right)^{\frac{1}{2}}$$

It is relatively easy to check that the above norms are indeed norms, except for the 2-form. In particular, the **triangle inequality** is not as easy to show [1].

[1] See Minkowski's Inequality.

Less obviously so, but true nonetheless, we can define the following p-norms on $\mathbb{K}^N$:

$$\left\| (x_n)_{n=1}^N \right\|_p := \left(\sum_{n=1}^N |x_n|^p \right)^{\frac{1}{p}} ,$$

for $1 \le p < \infty$.

Culture

Consider $V = \mathbb{M}_n(\mathbb{C})$, [2] *where $n \in \mathbb{N}$ is fixed. For $T \in \mathbb{M}_n(\mathbb{C})$, we define the* ***singular numbers*** *of T to be*

$$s_1(T) \ge s_2(T) \ge \ldots \ge s_n(T) \ge 0,$$

where $\sigma(T^*T) = \{s_1(T)^2, s_2(T)^2, \ldots, s_n(T)^2\}$, including multiplicity. Then we can define

$$\|T\|_p := \left(\sum_{i=1}^n s_i(T)^p \right)^{\frac{1}{p}}$$

for $1 \le p < \infty$, which is called the p-norm of T on $\mathbb{M}_n(\mathbb{C})$.

[2] Note that $\mathbb{M}_n(\mathbb{C})$ is the set of $n \times n$ matrices over $\mathbb{C}$.

Example 1.1.3

Let

$$V = \mathcal{C}([0,1], \mathbb{K}) = \{ f : [0,1] \to \mathbb{K} \mid f \text{ is continuous } \}.$$

Then

$$\|f\|_{\sup} := \sup\{|f(x)| \mid x \in [0,1]\}$$

[3] defines a norm on $\mathcal{C}([0,1], \mathbb{K})$.

[3] Some authors use $\|f\|_\infty$, but we will have the notation $\|[f]\|_\infty$ later on, and so we shall use $\|f\|_{\sup}$ for clarity.

A sequence $(f_n)_{n=1)^\infty}$ in V converges in this norm to some $f \in V$, i.e.

$$\lim_{n\to\infty} \|f_n - f\|_{\sup} = 0,$$

which means that $(f_n)_{n=1}^\infty$ converges uniformly to f on $[0,1]$.

Definition 2 (Normed Linear Space)

*A **normed linear space (NLS)** is a pair $(V, \|\cdot\|)$ where V is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is a norm on V.*

Definition 3 (Metric)

*Given an NLS $(V, \|\cdot\|)$, we can define a **metric** d on V (called the **metric induced by the norm**) as follows:*

$$d : V \times V \to \mathbb{R} \quad d(x,y) = \|x - y\|,$$

such that

- *$d(x,y) \geq 0$ for all $x, y \in V$ and $d(x,y) = 0 \iff x = y$;*
- *$d(x,y) = d(y,x)$; and*
- *$d(x,y) \leq d(x,z) + d(y,z)$.*

Note 1.1.2

Norms are all metrics, and so any space that has a norm will induce a metric on the space.

Definition 4 (Banach Space)

We say that an NLS $(V, \|\cdot\|)$ is ***complete*** *or is a* ***Banach Space*** *if the corresponding (V, d), where d is the metric induced by the norm, is complete* [4].

[4] Completeness of a metric space is such that any of its Cauchy sequences converges in the space.

Example 1.1.4

$(\mathcal{C}([0,1], \mathbb{K}), \|\cdot\|_{\sup})$ is a Banach space.

Example 1.1.5

We can define a 1-norm $\|\cdot\|_1$ on $\mathcal{C}([0,1], \mathbb{K})$ via

$$\|f\|_1 := \int_0^1 |f| \,.$$

Then $(\mathcal{C}([0,1], \mathbb{K}), \|\cdot\|_1)$ is an NLS.

Exercise 1.1.2

Show that $(\mathcal{C}([0,1], \mathbb{K}), \|\cdot\|_1)$ is not complete, which will then give us an example of a ***normed linear space that is not Banach****.*

Proof

Consider the sequence $(f_n)_{n=1}^{\infty}$ of continuous functions given by

$$f_n(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} \\ n\left(x + \frac{1}{2}\right) & \frac{1}{2} \leq x \leq \frac{1}{2} + \frac{1}{n} \\ 1 & \text{otherwise} \end{cases}$$

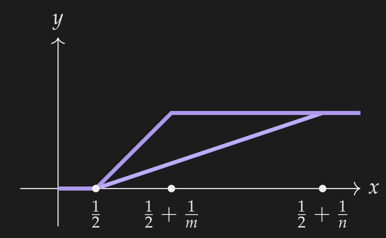

Figure 1.1: Sequence of functions $(f_n)_{n=1}^{\infty}$. We show for two indices $n < m$.

Note that the sequence $(f_n)_{n=1}^{\infty}$ is indeed **Cauchy**: let $\varepsilon > 0$ and $|n - m| < \frac{\varepsilon}{\left|x - \frac{1}{2}\right|}$, and then we have

$$\begin{aligned} |f_n(x) - f_m(x)| &= \left| n\left(x - \frac{1}{2}\right) - m\left(x - \frac{1}{2}\right) \right| \\ &= \left| (n - m)\left(x - \frac{1}{2}\right) \right| = |n - m| \left| x - \frac{1}{2} \right| < \varepsilon. \end{aligned}$$

However, it is clear that the sequence $(f_n)_{n=1}^{\infty}$ converges to the piecewise function (in particular, a non-continuous function)

$$f(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}.$$

□

Example 1.1.6

If $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ and $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ are NLS's, and if $T : \mathfrak{X} \to \mathfrak{Y}$ is a linear map, we define the **operator norm** of T to be

$$\|T\| := \sup\{\|T(x)\|_{\mathfrak{Y}} \mid \|x\|_{\mathfrak{X}} \leq 1\}.$$

We set

$$B(\mathfrak{X}, \mathfrak{Y}) := \{T : \mathfrak{X} \to \mathfrak{Y} \mid T \text{ is linear }, \|T\| < \infty\}.$$

Note that for any such linear map T, $\|T\| < \infty \iff T$ is continuous. Thus $B(\mathfrak{X}, \mathfrak{Y})$ is the set of all continuous functions from $\mathfrak{X}$ into $\mathfrak{Y}$.

Then $(B(\mathfrak{X}, \mathfrak{Y}), \|\cdot\|)$ is an NLS.

It is likely that we have seen this in Real Analysis.

Exercise 1.1.3

Show that $(B(\mathfrak{X}, \mathfrak{Y}), \|\cdot\|)$ is complete iff $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ is complete.

Note 1.1.3

*One example of the last example is when $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}}) = (\mathbb{K}, |\cdot|)$. In this case, $B(\mathfrak{X}, \mathbb{K})$ is known as the **dual space** of $\mathfrak{X}$, or simple the **dual** of $\mathfrak{X}$.*

We are interested in integrating over Banach spaces.

Definition 5 (Partition of a Set)

*Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space and $f : [a, b] \to \mathfrak{X}$ a function, where $a < b \in \mathbb{R}$. A **partition** P of $[a, b]$ is a finite set*

$$P = \{a = p_0 < p_1 < \ldots < p_N = b\}$$

for some $N \geq 1$. The set of all partitions of $[a, b]$ is denoted by $\mathcal{P}[a, b]$.

Definition 6 (Test Values)

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space and $f : [a, b] \to \mathfrak{X}$ a function, where $a < b \in \mathbb{R}$. Let $P \in \mathcal{P}[a, b]$. A set

$$P^* := \{p_k^*\}_{k=1}^N$$

satisfying

$$p_{k-1} \leq p_k^* \leq p_k, \text{ for } 1 \leq k \leq n$$

*is called a set of **test values** for P.*

Definition 7 (Riemann Sum)

*Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space and $f : [a, b] \to \mathfrak{X}$ a function, where $a < b \in \mathbb{R}$. Let $P \in \mathcal{P}[a, b]$ and P^* its corresponding set of test values. We define the **Riemann sum** as*

$$S(f, P, P^*) = \sum_{k=1}^N f(p_k^*)(p_k - p_{k-1}).$$

Remark 1.1.2

1. *Note that because Definition 5, $p_k - p_{k-1} > 0$.*
2. *When $(\mathfrak{X}, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, then this is the usual Riemann sum from first-year calculus.*
3. *In general, note that*

$$\frac{1}{b-a} S(f, P, P^*) = \sum_{k=1}^N \lambda_k f(p_k^*),$$

where $0 < \lambda_k = \frac{p_k - p_{k-1}}{b-a} < 1$ and [5]

$$\sum_{k=1}^N \lambda_k = 1.$$

[5] via the fact that the λ_k's form a telescoping sum

*So $\frac{1}{b-a}S(f, P, P^*)$ is an **averaging** of f over $[a, b]$. We call $\frac{1}{b-a}S(f, P, P^*)$ the **convex combination** of the $f(p_k^*)$'s.*

Example 1.1.7 (Silly example)

Let $(\mathfrak{X} = \mathcal{C}([-\pi, \pi], \mathbb{K}), \|\cdot\|_{\sup})$. Let

$$f : [0, 1] \to \mathfrak{X} \text{ such that } x \mapsto e^{2\pi x} \sin 7\theta + \cos x \cos(12\theta),$$

where $\theta \in [-\pi, \pi]$. Now if we consider the partition

$$P = \left\{ -\pi, \frac{1}{10}, \frac{1}{2}, \pi \right\}$$

and its corresponding test value

$$P^* = \left\{ 0, \frac{1}{3}, 2 \right\},$$

then

$$\begin{aligned} S(f, P, P^*) &= f(0) \left(\frac{1}{10} + \pi \right) + f\left(\frac{1}{3}\right) \left(\frac{1}{2} - \frac{1}{10} \right) + f(2) \left(\pi - \frac{1}{2} \right) \\ &= (\sin 7\theta + \cos 12\theta) \left(\pi + \frac{1}{10} \right) \\ &\quad + \left(e^{\frac{2\pi}{3}} \sin 7\theta + \cos \frac{1}{3} \cos 12\theta \right) \left(\frac{2}{5} \right) \\ &\quad + (e^{4\pi} \sin 7\theta + \cos 2 \cos 12\theta) \left(\pi - \frac{1}{2} \right) \end{aligned}$$

Definition 8 (Refinement of a Partition)

*Let $a < b \in \mathbb{R}$, and $P \in \mathcal{P}[a, b]$. We say Q is a **refinement** of P is $Q \in \mathcal{P}[a, b]$ and $P \subseteq Q$.*

Note 1.1.4

In simpler words, Q is a "finer" partition that is based on P.

Definition 9 (Riemann Integrable)

Let $a < b \in \mathbb{R}$, $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space and $f : [a, b] \to \mathfrak{X}$ be a function. We say that f is ***Riemann integrable*** *over $[a, b]$ if $\exists x_0 \in \mathfrak{X}$ such that*

$$\forall \varepsilon > 0 \quad \exists P \in \mathcal{P}[a, b],$$

such that if Q is any refinement of P, and Q^* is any set of test values of Q, then

$$\|x_0 - S(f, Q, Q^*)\|_{\mathfrak{X}} < \varepsilon.$$

In this case, we write

$$\int_a^b f = x_0.$$

Proposition 1 (Uniqueness of the Riemann Integral)

If f is Riemann integrable over $[a, b]$, then the value of $\int_a^b f$ is unique.

Proof

Suppose not, i.e.

$$\int_a^b f = x_0 \text{ and } \int_a^b f = y_0$$

for some $x_0 \neq y_0$. Then, let

$$\varepsilon = \frac{\|x_0 - y_0\|}{2},$$

which is > 0 since $\|x_0 - y_0\| > 0$. Let $P_{x_0}, P_{y_0} \in \mathcal{P}[a, b]$ be partitions corresponding to x_0 and y_0 as in the definition of Riemann integrability.

Then, let $R = P_{x_0} \cup P_{y_0}$, so that R is a **common refinement** of P_{x_0} and P_{y_0}. If Q is any refinement of R, then Q is also a common refinement of P_{x_0} and P_{y_0}. Then for any test values Q^* of Q, we have

$$\begin{aligned} 2\varepsilon &= \|x_0 - y_0\| \\ &\leq \|x_0 - S(f, Q, Q^*)\| + \|S(f, Q, Q^*) - y_0\| < \varepsilon + \varepsilon = 2\varepsilon, \end{aligned}$$

which is a contradiction.

Thus $x_0 = y_0$ as required. □

Theorem 2 (Cauchy Criterion of Riemann Integrability)

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space, $a < b \in \mathbb{R}$ and $f : [a,b] \to \mathfrak{X}$ be a function. TFAE:

1. *f is Riemann integrable over $[a,b]$;*
2. *$\forall \varepsilon > 0$, $R \in \mathcal{P}[a,b]$, if P, Q is any refinement of R, and P^* (respectively Q^*) is any test values of P (respectively Q), then*

$$\|S(f,P,P^*) - S(f,Q,Q^*)\|_{\mathfrak{X}} < \varepsilon.$$

Proof

($\implies$) This is a rather straightforward proof. Suppose $P, Q \in \mathcal{P}[a,b]$ is some refinement of the given partition $R \in \mathcal{P}[a,b]$, and P^*, Q^* any test values for P, Q, respectively. Then by assumption and Proposition 1, $\exists x_0 \in \mathfrak{X}$ such that

$$\|x_0 - S(f,P,P^*)\|_{\mathfrak{X}} < \frac{\varepsilon}{2} \text{ and } \|x_0 - S(f,Q,Q^*)\|_{\mathfrak{X}} < \frac{\varepsilon}{2}.$$

It follows that

$$\begin{aligned} &\|S(f,P,P^*) - S(f,Q,Q^*)\|_{\mathfrak{X}} \\ &\leq \|x_0 - S(f,P,P^*)\|_{\mathfrak{X}} + \|x_0 - S(f,Q,Q^*)\|_{\mathfrak{X}} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

($\impliedby$) By hypothesis, wma $\varepsilon = \frac{1}{n}$ for some $n \geq 1$, such that if P, Q are any refinements of the partition $R_n \in \mathcal{P}[a,b]$, and P^*, Q^* are the respective arbitrary test values, then

$$\|S(f,P,P^*) - S(f,Q,Q^*)\|_{\mathfrak{X}} < \frac{1}{n}$$

Now for each $n \geq 1$, define

$$W_n := \bigcup_{k=1}^{n} R_k \in \mathcal{P}[a,b],$$

so that W_n is a common refinement for $R_1, R_2, \ldots, R_n$. For each $n \geq 1$, let W_n^* be an arbitrary set of test values for W_n. For simplicity, let us write

$$x_n = S(f,\, W_n,\, W_n^*), \text{ for each } n \geq 1.$$

[6]

Claim: $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence If $n_1 \geq n_2 > N \in \mathbb{N}$, then

$$\|x_{n_1} - x_{n_2}\|_{\mathfrak{X}} = \left\|S(f, W_{n_1}, W_{n_1}^*) - S(f, W_{n_2}, W_{n_2}^*)\right\| < \frac{1}{N}$$

by our assumption, since W_{n_1}, W_{n_2} are refinements of R_N. Then by picking $N = \frac{1}{\varepsilon}$ for any $\varepsilon > 0$, we have that $(x_n)_{n=1}^{\infty}$ is indeed a Cauchy sequence in $\mathfrak{X}$.

[6] Note that it would be nice if for the finer and finer partitions that we have constructed, i.e. the W_n's, give us a convergent sequence of Riemann sums, since it makes sense that this convergence will give us the final value that we want.

Since $\mathfrak{X}$ is a Banach space, it is complete, and so $\exists x_0 := \lim_{n\to\infty} x_n \in \mathfrak{X}$. It remains to show that, indeed,

$$x_0 = \int_a^b f.$$

Let $\varepsilon > 0$, and choose $N \geq 1$ such that

- $\frac{1}{N} < \frac{\varepsilon}{2}$; and
- $k \geq N$ implies that $\|x_k - x_0\| < \frac{\varepsilon}{2}$.

Then suppose that V is any refinement of W_N, and V^* is an arbitrary set of test values of V. Then we have

$$\begin{aligned}\|x_0 - S(f,V,V^*)\|_{\mathfrak{X}} &\leq \|x_0 - x_N\|_{\mathfrak{X}} + \|x_N - S(f,V,V^*)\|_{\mathfrak{X}} \\ &< \frac{\varepsilon}{2} + \|S(f,W_N,W_N^*) - S(f,V,V^*)\|_{\mathfrak{X}} \\ &< \frac{\varepsilon}{2} + \frac{1}{N} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

It follows that

$$\int_a^b f = x_0,$$

as desired. $\square$

In first-year calculus, all continuous functions over $\mathbb{R}$ are integrable. A similar result holds in Banach spaces as well. In the next lecture, we shall prove the following theorem.

Theorem (Continuous Functions are Riemann Integrable)

Let $(\mathfrak{X}, \|\cdot\|)$ *be a Banach space and* $a < b \in \mathbb{R}$. *If* $f : [a, b] \to \mathfrak{X}$ *is continuous, then* f *is Riemann integrable over* $[a, b]$.

2 Lecture 2 May 9th 2019

2.1 Riemannian Integration (Continued)

We shall now prove the last theorem stated in class.

Theorem 3 (Continuous Functions are Riemann Integrable)

Let $(\mathfrak{X}, \|\cdot\|)$ be a Banach space and $a < b \in \mathbb{R}$. If $f : [a, b] \to \mathfrak{X}$ is continuous, then f is Riemann integrable over $[a, b]$.

Strategy

This is rather routine should one have gone through a few courses on analysis, and especially on introductory courses that involves Riemannian integration.

We shall show that if $P_N \in \mathcal{P}[a, b]$ is a partition of $[a, b]$ into 2^N subintervals of equal length $\frac{b-a}{2^N}$, and if we use $P_N^* = P_n \setminus \{a\}$ as the set of test values for P_N, which consists of the right-endpoints of each the subintervals in P_N, then the sequence $(S(f, P_N, P_N^*))_{N=1}^{\infty}$ converges in $\mathfrak{X}$ to $\int_a^b f$.

Note that this choice of partition is a valid move, since any of these P_N's, for different N's, is a refinement of some other partition of $[a, b]$, and if we choose a different set of test values, then we may as well consider an even finer partition.

Proof

First, note that since $[a, b]$ is closed and bounded in $\mathbb{R}$, it is compact.

Also, we have that X is a metric space (via the metric induced by the norm). This means that **any continuous function f on $[a, b]$ is uniformly continuous on $[a, b]$**. In other words,

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, y \in [a, b]$$
$$|x - y| < \delta \implies \|f(x) - f(y)\| < \frac{\varepsilon}{2(b-a)}.$$

Claim: $(S(f, P_N, P_N^*))_{N=1}^{\infty}$ is Cauchy Now by picking $P_N \in \mathcal{P}[a, b]$ and set of test values P_N^* as described in the strategy above, we proceed by picking $M > 0$ such that $\frac{b-a}{2^M} < \delta$. Then for any $K \geq L \geq M$, since each of the subintervals have length $\frac{b-a}{2^L}$ and $\frac{b-a}{2^K}$ for P_L and P_K respectively, if we write

$$P_L = \{a = p_0 < p_1 < \ldots < p_{2^L} = b\}$$

and

$$P_K = \{a = q_0 \leq q_1 < \ldots < q_{2^K} = b\},$$

then $p_j = q_{j 2^{K-L}}$ [1] for all $0 \leq j \leq 2^L$. By uniform continuity, for $1 \leq j \leq 2^L$, wma

[1] This is not immediately clear on first read. Think of a as 0.

$$\left\| f(p_j^*) - f(q_s^*) \right\| < \frac{\varepsilon}{2(b-a)}, \text{ where } (j-1)2^{K-L} < s \leq j 2^{K-L}.$$

We can see that

$$\begin{aligned}
&\|S(f, P_L, P_L^*) - S(f, P_K, P_K^*)\| \\
&= \left\| \sum_{j=1}^{2^L} \sum_{s=(j-1)2^{K-L}+1}^{j2^{K-L}} (f(p_j) - f(q_s))(q_s - q_{s-1}) \right\| \\
&\leq \sum_{j=1}^{2^L} \sum_{s=(j-1)2^{K-L}+1}^{j2^{K-L}} \left\| f(p_j) - f(q_s) \right\| (q_s - q_{s-1}) \\
&\leq \sum_{j=1}^{2^L} \sum_{s=(j-1)2^{K-L}+1}^{j2^{K-L}} \frac{\varepsilon}{b-a} (q_s - q_{s-1}) \\
&= \frac{\varepsilon}{b-a} \sum_{s=1}^{2^K} (q_s - q_{s-1}) \\
&= \frac{\varepsilon}{2(b-a)} (b-a) = \frac{\varepsilon}{2}.
\end{aligned}$$

This proves our claim.

Since $\mathfrak{X}$ is a Banach space, and hence complete, we have that the sequence $(S(f, P_N, P_N^*))_{N=1}^{\infty}$ has a limit $x_0 \in \mathfrak{X}$.

It remains to show that $\int_a^b f = x_0$. [2]

[2] The rest of this proof is similar to the above proof.

Let $\varepsilon > 0$, and choose $T \geq 1$ such that $\frac{b-a}{2^T} < \delta$ [3], so that we have

[3] Note that this is still the same δ as in the first δ in this entire proof.

$$\|x_0 - S(f, P_T, P_T^*)\| < \frac{\varepsilon}{2}.$$

Now let $R = \{a = r_0 < r_1 < \ldots < r_J = b\} \in \mathcal{P}[a, b]$ such that $P_T \subseteq R$. Then there exists a sequence

$$0 = j_0 < j_1 < \ldots < j_{2^T} = J$$

such that

$$r_{j_k} = p_k, \text{ where } 0 \leq k \leq 2^T.$$

Let R^* be any set of test values of R. Note that for $j_{k-1} \leq s \leq j_k$, it is clear that

$$|p_k^* - r_s^*| \leq |p_k - p_{k-1}| = \frac{b-a}{2^T} < \delta.$$

Thus

$$\begin{aligned}
&\|S(f, P_T, P_T^*) - S(f, R, R^*)\| \\
&\leq \sum_{k=1}^{2^T} \sum_{s_{j_{k-1}+1}}^{j_k} \|f(p_k^*) - f(r_s^*)\| (r_s - r_{s-1}) \\
&< \frac{\varepsilon}{2(b-a)} \sum_{k=1}^{2^T} \sum_{s_{j_{k-1}+1}}^{j_k} (r_s - r_{s-1}) \\
&= \frac{\varepsilon}{2(b-a)} (b-a) = \frac{\varepsilon}{2}.
\end{aligned}$$

Putting everything together, we have

$$\begin{aligned}
&\|x_0 - S(f, R, R^*)\| \\
&\leq \|x_0 - S(f, P_T, P_T^*)\| + \|S(f, P_T, P_T^*) - S(f, R, R^*)\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

We can also find another refinement of P_T, say Q, that works similarly as in the case of R. It follows from Theorem 2 that

$$x_0 = \int_a^b f,$$

i.e. that f is indeed Riemann integrable over $[a, b]$. □

The following is a corollary whose proof shall be left as an exercise.

Corollary 4 (Piecewise Functions are Riemann Integrable)

*A **piecewise continuous** function is also Riemann integrable: if $f : [a, b] \to \mathfrak{X}$ is piecewise continuous, then f is Riemann integrable.*

Exercise 2.1.1

Prove Corollary 4.

Let us exhibit a function that is not Riemann integrable.

Definition 10 (Characteristic Function)

*Given a subset E of a set $\mathbb{R}$, we define the **characteristic function** of E as a function $\chi_E : \mathbb{R} \to \mathbb{R}$ given by*

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

Example 2.1.1

Consider the set $E = \mathbb{Q} \cap [0, 1] \subseteq \mathbb{R}$. Let $P \in \mathcal{P}[0, 1]$ such that

$$P = \{0 = p_0 < p_1 < \ldots < p_N = 1\},$$

and let

$$P^* = \{p_k^*\}_{k=1}^N \text{ and } P^{**} = \{p_k^{**}\}_{k=1}^N$$

be 2 sets of test values for P, such that we have

$$p_k^* \in \mathbb{Q} \text{ and } p_k^{**} \in \mathbb{R} \setminus \mathbb{Q}.$$

Then we have

$$\begin{aligned} S(\chi_E, P, P^*) &= \sum_{k=1}^{N} \chi_E(p_k^*)(p_k - p_{k-1}) \\ &= \sum_{k=1}^{N} 1 \cdot (p_k - p_{k-1}) \\ &= p_N - p_0 = 1 - 0 = 1, \end{aligned}$$

and

$$\begin{aligned} S(\chi_E, P, P^{**}) &= \sum_{k=1}^{N} \chi_E(p_k^{**})(p_k - p_{k-1}) \\ &= \sum_{k=1}^{N} 0 \cdot (p_k - p_{k-1}) \\ &= 0. \end{aligned}$$

It is clear that the Cauchy criterion fails for χ_E. This shows that χ_E is not Riemann integrable.

Remark 2.1.1

Let us once again consider $E = \mathbb{Q} \cap [0,1]$. Note that E is ***denumerable*** [4]. *We may thus write*

$$E = \{q_n\}_{n=1}^{\infty}.$$

Now, for $k \geq 1$, define

$$f_k(x) = \sum_{n=1}^{k} \chi_{\{q_n\}}(x).$$

In other words, $f_k = \chi_{\{q_1,\ldots,q_k\}}$. Furthermore, we have that

$$f_1 \leq f_2 \leq f_3 \ldots \leq \chi_E.$$

Moreover, we have that $\forall x \in [0,1]$,

$$\chi_E(x) = \lim_{k\to\infty} f_k(x),$$

[4] This means that E is countably infinite.

and

$$\int_0^1 f_k = 0 \textit{ for all } k \geq 1.$$

And yet, we have that $\int_0^1 \chi_E$ does not exist!

We want to develop a different integral that will 'cover' for this 'pathological' behavior of where the Riemann integral fails.

The rough idea is as follows.

In Riemann integration, when integrating over an interval $[a, b]$, we partitioned $[a, b]$ into subintervals. This happens on the x-axis.

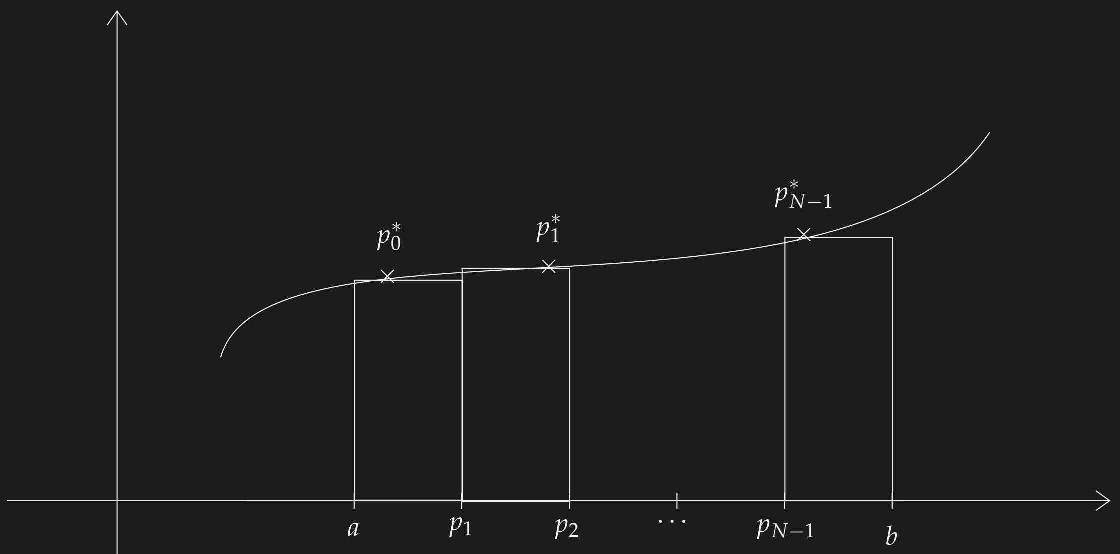

Figure 2.1: Rough illustration of how Riemann's integration works

In each of the subintervals of the partition, we pick out a **test value** p_i^*, and basically draw a rectangle with base at $[p_i, p_{i+1}]$ and height from 0 to p_i^*.

What we shall do now is that we **partition the range of f on the y-axis**, instead of the x-axis as we do in Riemannian integration.

In particular, given a function $f : [a, b] \to \mathbb{R}$, we first partition the range of f into subintervals $[y_{k-1}, y_k]$, where $1 \leq k \leq N$. Then, we set

$$E_k = \{x \in [a, b] : f(x) \in [y_{k-1}, y_k]\} \text{ for } 1 \leq k \leq N.$$

This will then allow us to estimate the integral of f over $[a, b]$ by the

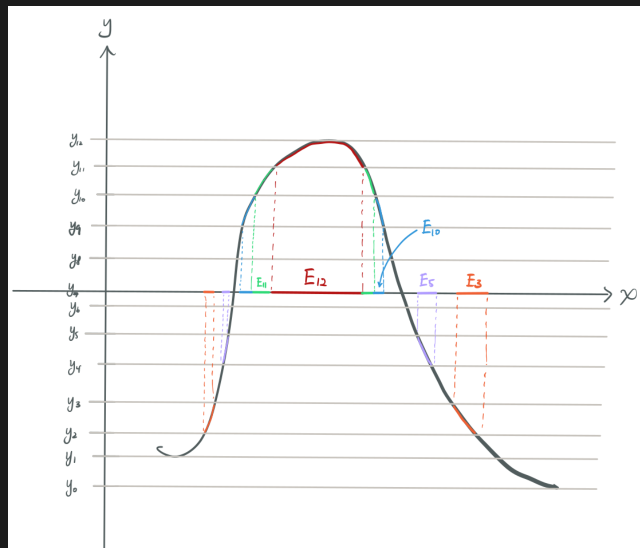

Figure 2.2: A sketch of what's happening with the construction of the E_k's

expression

$$\sum_{k=1}^{N} y_k m E_k,$$

where each of the $y_k m E_k$ are called **simple functions**. In the expression, mE_k denotes a "measure" [5] of E_k.

[5] Note that a measure is simply a generalization of the notion of 'length'.

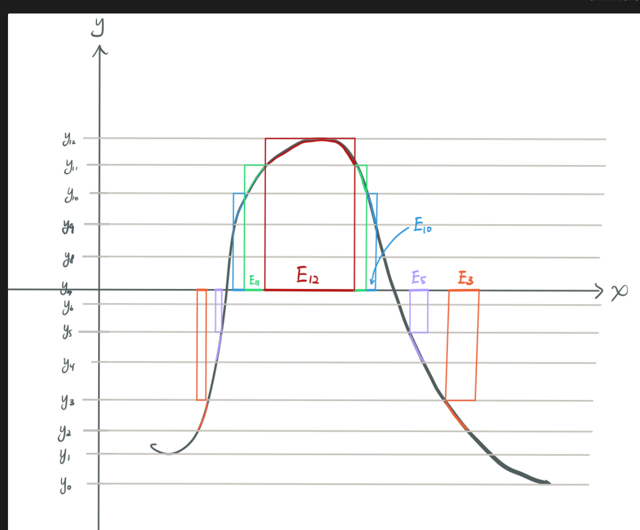

Figure 2.3: Drawing out the rectangles of $y_k m E_k$ from Figure 2.2.

We observe that E_k need not be a particularly well-behaved set. However, note that we may rearrange the possibly scattered pieces of each E_k together, so as to form a 'continuous' base for the rectangle. We need our definition of a measure to be able to capture this.

The following is an analogy from Lebesgue himself on comparing

Lebesgue integration and Riemann integration [6]:

> *I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral.*
> *But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral.*

[6] Siegmund-Schultze, R. (2008). *Henri Lesbesgue, in Timothy Gowers, June Barrow-Green, Imre Leader (eds.), Princeton Companion to Mathematics*. Princeton University Press

The insight here is that one can freely arrange the values of the functions, all the while preserving the value of the integral.

- This requires us to have a better understanding of what a measure is.
- This process of rearrangement converts certain functions which are extremely difficult to deal with, or outright impossible, with the Riemann integral, into easily digestible pieces using Lebesgue integral.

2.2 *Lebesgue Outer Measure*

Goals of the section

1. Define a "measure of length" on as many subsets of $\mathbb{R}$ as possible.
2. The definition should agree with our intuition of what a 'length' is.

Definition 11 (Length)

*For $a \leq b \in \mathbb{R}$, we define the **length** of the interval (a, b) to be $b - a$, and we write*

$$\ell((a, b)) := b - a.$$

We also define

- $\ell(\emptyset) = 0$; *and*
- $\ell((a, \infty)) = \ell((-\infty, b)) = \ell((-\infty, \infty)) = \infty$.

Definition 12 (Cover by Open Intervals)

Let $E \subseteq \mathbb{R}$. A countable collection $\{I_n\}_{n=1}^{\infty}$ of open intervals is said to be a ***cover of E by open intervals*** *if $E \subseteq \bigcup_{n=1}^{\infty} I_n$.*

Note 2.2.1

In this course, the only covers that we shall use are open intervals, and so we shall henceforth refer to the above simply as covers of E.

Before giving what immediately follows from the above, I shall present the following notion of an outer measure.

Definition 13 (Outer Measure)

Let $\emptyset \neq X$ be a set. An ***outer measure*** *μ on X is a function*

$$\mu : \mathcal{P}(X) \to [0, \infty] := [0, \infty) \cup \{\infty\}$$

which satisfies

1. $\mu\emptyset = 0$;
2. (***monotone increment*** or ***monotonicity***) $E \subseteq F \subseteq X \implies \mu E \leq \mu F$; and
3. (***countable subadditivity*** or ***σ-subadditivity***) $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu E_n.$$

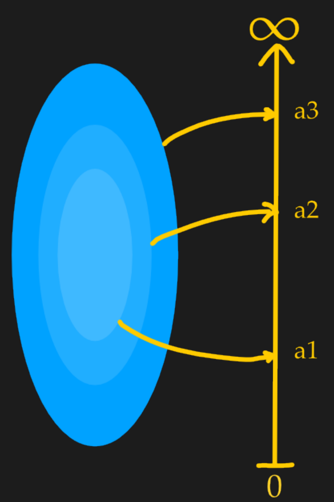

Figure 2.4: Idea of the outer measure

Note 2.2.2

Note that by the monotonicity, the σ-subadditivity condition is equivalent *to: given* $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$ *and* $F \subseteq \bigcup_{n=1}^{\infty} E_n$*, we have that*

$$\mu(F) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

Definition 14 (Lebesgue Outer Measure)

We define the ***Lebesgue outer measure*** *as a function* $m^* : \mathcal{P}(X) \to \mathbb{R}$ *such that*

$$m^* E := \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : E \subseteq \bigcup_{n=1}^{\infty} I_n \right\}.$$

We cheated a little bit by calling the above an outer measure, so let us now justify our cheating.

Proposition 5 (Validity of the Lebesgue Outer Measure)

m^* *is indeed an outer measure.*

Proof

$\mu\emptyset = 0$ We consider a sequence of sets $\{I_n\}_{n=1}^{\infty}$ such that $I_n = \emptyset$ for each $n = 1, \ldots, \infty$. It is clear that $\emptyset \subseteq \bigcup_{n=1}^{\infty} I_n$. Also, we have that $\ell(I_n) = 0$ for all $n = 1, \ldots, \infty$. It follows that

$$0 \leq m^*(\emptyset) \leq \sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} 0 = 0,$$

where the inequality is simply by the definition of m^* being an infimum, **not to be confused with σ-subadditivity**. We thus have that

$$m^*(\emptyset) = 0.$$

Monotonicity Suppose $E \subseteq F \subseteq \mathbb{R}$, and $\{I_n\}_{n=1}^{\infty}$ a cover of F. Then

$$E \subseteq F \subseteq \bigcup_{n=1}^{\infty} I_n.$$

In particular, all covers of F are also covers of E, i.e.

$$\left\{ \{J_m\}_{m=1}^{\infty} : E \subseteq \bigcup_{m=1}^{\infty} J_m \right\} \subseteq \left\{ \{I_n\}_{n=1}^{\infty} : F \subseteq \bigcup_{n=1}^{\infty} I_n \right\}.$$

It follows that

$$m^* E \leq m^* F.$$

σ-subaddivitity Consider $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$ such that $E \subseteq \bigcup_{n=1}^{\infty} E_n$. WTS

$$m^* E \leq \sum_{n=1}^{\infty} m^* E_n.$$

Now if the sum of the RHS is infinite, i.e. if any of the $m^* E_n$ is infinite, then the inequality comes for free. Thus WMA $\sum_{n=1}^{\infty} E_n < \infty$, and in particular that $m^* E_n < \infty$ for all $n = 1, \ldots, \infty$.

To do this, let $\varepsilon > 0$. Since $m^* E_n < \infty$ for all n, we can find covers $\left\{ I_k^{(n)} \right\}_{k=1}^{\infty}$ for each of the E_n's such that

$$\sum_{k=1}^{\infty} \ell\left(I_k^{(n)} \right) < m^* E_n + \frac{\varepsilon}{2^n}.$$

Then, we have that

$$E \subseteq \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} I_k^{(n)}.$$

Then by $m^* E$ being the infimum of the sum of lengths of the covering intervals, we have that

$$\begin{aligned} m^* E &\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \ell\left(I_k^{(n)} \right) \\ &\leq \sum_{n=1}^{\infty} \left(m^* E_n + \frac{\varepsilon}{2^n} \right) \\ &= \sum_{n=1}^{\infty} m^* E_n + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} \\ &= \sum_{n=1}^{\infty} m^* E_n + \varepsilon. \end{aligned}$$

Since ε was arbitrary, we have that

$$m^* E_n \leq \sum_{n=1}^{\infty} m^* E_n,$$

as desired. □

Corollary 6 (Lebesgue Outer Measure of Countable Sets is Zero)

If $E \subseteq \mathbb{R}$ is countable, then $m^* E = 0$.

Proof

We shall prove for when E is denumerable, for the finite case follows a similar proof. Let us write $E = \{x_n\}_{n=1}^{\infty}$. Let $\varepsilon > 0$ and

$$I_n = \left(x_n - \frac{\varepsilon}{2^{n+1}}, \, x_n + \frac{\varepsilon}{2^{n+1}} \right).$$

Then it is clear that $\{I_n\}_{n=1}^{\infty}$ is a cover of E.

It follows that

$$0 \leq m^* E \leq \sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Thus as $\varepsilon \to 0$, we have that

$$m^* E = 0,$$

as expected. □

Corollary 7 (Lebesgue Outer Measure of $\mathbb{Q}$ is Zero)

We have that $m^* \mathbb{Q} = 0$.

In the proofs above that we have looked into, and based on the intuitive notion of the length of an open interval, it is compelling to simply conclude that

$$m^*(a, b) = \ell(a, b) = b - a.$$

However, looking back at Definition 14, we know that that is not how $m^*(a, b)$ is defined.

This leaves us with an interesting question:

> *how does our notion of measure $m^*(a, b)$ of an interval compare with the notion of the length of an interval?*

By taking $I_1 = (a, b)$ and $I_n = \emptyset$ for $n \geq 2$, it is rather clear that $\{I_n\}_{n=1}^{\infty}$ is a cover of (a, b), and so we have

$$m^*(a, b) \leq \ell(a, b) = b - a. \tag{2.1}$$

However, the other side of the game is not as easy to confirm: we would have to consider **all** possible covers of (a, b), which is a lot.

Another question that we can ask ourselves seeing Equation (2.1) is why can't $m^*(a, b)$ be something that is strictly less than the length to give us an even more 'precise' measurement?

To answer these questions, it is useful to first consider the outer measure of a closed and bounded interval, e.g. $[a, b]$, since these intervals are **compact** under the **Heine-Borel Theorem**. This will give us a finite subcover for every infinite cover of the compact interval, which is easy to deal with.

We shall see that with the realization of the outer measure of a compact interval, we will also be able to find the outer measure of intervals that are neither open nor closed.

We shall prove the following proposition in the next lecture. Note that for the sake of presentation, I shall abbreviate the Lebesgue Outer Measure as LOM.

Proposition (LOM of Arbitrary Intervals)

Suppose $a < b \in \mathbb{R}$. Then

1. $m^*([a, b]) = b - a$; *and therefore*
2. $m^*((a, b]) = m^*([a, b)) = m^*((a, b)) = b - a$.

3 Lecture 3 May 14th 2019

3.1 Lebesgue Outer Measure Continued

Proposition 8 (LOM of Arbitrary Intervals)

Suppose $a < b \in \mathbb{R}$. Then

1. $m^*([a,b]) = b - a$; *and therefore*
2. $m^*((a,b]) = m^*([a,b)) = m^*((a,b)) = b - a$.

Proof

1. Consider $a < b \in \mathbb{R}$. Let $\varepsilon > 0$, and let

$$I_1 = \left(a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right)$$

and $I_n = \emptyset$ for $n \geq 2$. Then $\{I_n\}_{n=1}^{\infty}$ is a cover of $[a,b]$. This means that

$$m^*([a,b]) \leq \sum_{n=1}^{\infty} \ell(I_n) = b - a + \varepsilon.$$

So for all $\varepsilon \to 0$, we have that

$$m^*([a,b]) \leq b - a.$$

[1] Conversely, if $[a,b]$ is covered by open intervals $\{I_n\}_{n=1}^{\infty}$, then by compactness of $[a,b]$ (via the **Heine-Borel Theorem**), we know that we can cover $[a,b]$ by finitely many of these intervals, and let us denote these as $\{I_n\}_{n=1}^{N}$, for some $1 \leq N < \infty$.

[1] For the converse, we know that $m^*([a,b]) = \inf ★$, where ★ is just a placeholder for you-know-what. So $m^*([a,b])$ is one of the sums. So if we can show that for an arbitrary sum, $\geq$ holds, our work is done.

WTS

$$\sum_{n=1}^{N} \ell(I_n) \geq b - a.$$

If LHS $= \infty$, then our work is done. Thus wlog, WMA each $I_n = (a_n, b_n)$ is a finite interval. Note that we have

$$[a, b] \subseteq \bigcup_{n=1}^{N} (a_n, b_n).$$

In particular, $a \in \bigcup_{n=1}^{N} I_n$. Thus, $\exists 1 \leq n_2 \leq N$ such that $a \in I_{n_1}$. Now if $b_{n_1} > b$, we shall stop this process for our work is done, since then $[a, b] \subseteq I_{n_1}$. Otherwise, if $b_{n_1} \leq b$, then $b_{n_1} \in [a, b] \subseteq \bigcup_{n=1}^{N} I_n$, which means that $\exists 1 \leq n_2 \leq N$ such that $b_{n_1} \in I_{n_2}$.

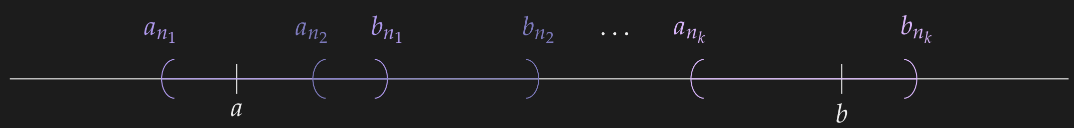

Figure 3.1: Our continual picking of $I_{n_1}, I_{n_2}, \ldots, I_{n_k}$

Notice that $n_1 \neq n_2$, since $b_{n_1} \notin I_{n_1}$ but $b_{n_1} \in I_{n_2}$.

Now once again, if $b_{n_2} > b$, then we shall stop this process since our work is done. Otherwise, we have $a < b_{n_2} \leq b$, and so $\exists 1 \leq n_3 \leq N$, $n_3 \neq n_1, n_2$, such that $b_{n_2} \in I_3$...

We continue with the above process for as long as $b_{n_k} \leq b$. We can thus find, for each k, $I_{n_{k+1}}$, where $n_{k+1} \in \{1, \ldots, N\} \setminus \{n_1, n_2, \ldots, n_k\}$, such that $b_{n_k} \in I_{n_{k+1}}$.

However, since each of the I_{n_k}'s are different, and since we only have N such intervals, there must exists a $K \leq N$ such that

$$b_{n_{K-1}} \leq b \text{ and } b_{n_K} > b.$$

It now suffices for us to show that

$$\sum_{j=1}^{K} \ell(I_{n_j}) \geq b - a.$$

Observe that

$$\sum_{j=1}^{K} \ell(I_{n_j}) = (b_{n_K} - a_{n_K}) + (b_{n_{K-1}} - a_{n_{K-1}}) + \ldots$$

$$
\begin{aligned}
&+ (b_{n_2} - a_{n_2}) + (b_{n_1} - a_{n_1}) \\
&= b_{n_K} + \underbrace{(b_{n_{K-1}} - a_{n_K})}_{\geq 0} + \underbrace{(b_{n_{K-2}} - a_{n_{K-1}})}_{\geq 0} + \ldots \\
&+ \underbrace{(b_{n_1} - a_{n_2})}_{\geq 0} - a_{n_1} \\
&\geq b_{n_K} - a_{n_1} \geq b - a.
\end{aligned}
$$

Thus

$$\sum_{n=1}^{\infty} \ell(I_n) \geq \sum_{n=1}^{N} \ell(I_n) \geq \sum_{j=1}^{K} \ell(I_{n_j}) \geq b - a,$$

whence

$$m^*([a, b]) \geq b - a.$$

It follows that, indeed,

$$m^*([a, b]) = b - a.$$

2. First, note that

$$m^*((a, b)) \leq m^*([a, b]) \leq b - a.$$

On the other hand, notice that $\forall 0 < \varepsilon < \frac{b-a}{2}$, we have that

$$[a + \varepsilon, b - \varepsilon] \subseteq (a, b),$$

and so by monotonicity,

$$(b - a) - 2\varepsilon = m^*([a + \varepsilon, b - \varepsilon]) \leq m^*((a, b)).$$

As $\varepsilon \to 0$, we have that

$$b - a \leq m^*((a, b)) \leq b - a.$$

So

$$m^*((a, b)) = b - a$$

as desired.

Finally, we have that

$$b - a = m^*((a, b)) \leq m^*((a, b]) \leq m^*([a, b]) = b - a,$$

and similarly

$$b - a = m^*((a,b)) \leq m^*([a,b)) \leq m^*([a,b]) = b - a.$$

Thus

$$m^*((a,b)) = m^*((a,b]) = m^*([a,b)) = b - a$$

as required. □

Proposition 9 (LOM of Infinite Intervals)

We have that $\forall a, b \in \mathbb{R}$,

$$\begin{aligned} m^*((a,\infty)) &= m^*([a,\infty)) \\ &= m^*((-\infty,b)) = m^*((-\infty,b]) \\ &= m^*\mathbb{R} = \infty. \end{aligned}$$

Proof

Observe that

$$(a, a+n) \subseteq (a, \infty)$$

for all $n \geq 1$. Thus

$$n = m^*((a, a+n)) \leq m^*((a,\infty))$$

for all $n \geq 1$. Hence

$$m^*((a,\infty)) = \infty$$

by definition.

All other cases follow similarly. □

Corollary 10 (Uncountability of $\mathbb{R}$)

$\mathbb{R}$ *is uncountable.*

Proof

We have that

$$m^*\mathbb{R} = \infty \neq 0,$$

and so it follows from Corollary 6, we must have that $\mathbb{R}$ is uncountable. □

Definition 15 (Translation Invariant)

*Let μ be an outer measure on $\mathbb{R}$. We say that μ is **translation invariant** if $\forall E \subseteq \mathbb{R}$,*

$$\mu(E) = \mu(E + \kappa)$$

for all $\kappa \in \mathbb{R}$, where

$$E + \kappa := \{x + \kappa : x \in E\}.$$

Proposition 11 (Translation Invariance of the LOM)

The Lebesgue outer measure is translation invariant.

Proof

Let $E \subseteq \mathbb{R}$ and $\kappa \in \mathbb{R}$. Note that E is covered by open intervals $\{I_n\}_{n=1}^{\infty}$ iff $E + \kappa$ is covered by $\{I_n + \kappa\}_{n=1}^{\infty}$.

Claim: $\forall n \geq 1$, $\ell(I_n + \kappa) = \ell(I_n)$ Write

$$I_n = (a_n, b_n).$$

Then

$$I_n + \kappa = (a_n + \kappa, b_n + \kappa).$$

Observe that

$$\ell(I_n + \kappa) = b_n + \kappa - (a_n - \kappa) = b_n - a_n = \ell(I_n),$$

as claimed. ⊣

By the claim, it follows that

$$\begin{aligned} m^*(E) &= \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : E \subseteq \bigcup_{n=1}^{\infty} \right\} \\ &= \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n + \kappa) : E + \kappa \subseteq \bigcup_{n=1}^{\infty} (I_n + \kappa) \right\} \\ &= m^*(E + \kappa). \end{aligned}$$

□

Remark 3.1.1

Suppose $E \subseteq \mathbb{R}$ and $E = \bigcup_{n=1}^{\infty} E_n$, where

$$E_i \cap E_j = \emptyset \text{ if } i \neq j.$$

Now by σ-subadditivity of m^*, we have that

$$m^* E \leq \sum_{n=1}^{\infty} m^* E_n.$$

However, equality is not guaranteed. Consider the following case: if $E = [0,1]$, we may have $E_n = [0,1]$ for all $n >= 1$, in which case $E = \bigcup_{n=1}^{\infty} E_n = [0,1]$, but

$$m^* E = m^*[0,1] = 1 < \infty = \sum_{n=1}^{\infty} m^* E_n.$$

It would be desirable to have

$$m^* E = \sum_{n=1}^{\infty} m^* E_n,$$

when the E_i's are pairwise disjoint, i.e. $E = \dot{\bigcup}_{n=1}^{\infty} E_n$. In fact, this would agree with our intuition, that if the outer measure is going to be our 'length'. Consider the example $A = [0,2] \cup [5,7]$. Then we would expect $m^* A = 2 + 2 = 4$.

However, this is actually impossible for an arbitrary number of collections.

Theorem 12 (Non-existence of a sensible Translation Invariant Outer Measure that is also σ-additive)

There does not exist a translation-invariant outer measure μ on $\mathbb{R}$ that satisfies

1. $\mu(\mathbb{R}) > 0$;
2. $\mu[0,1] < \infty$; *and*
3. μ *is* ***σ-additive****; i.e. if* $\{E_n\}_{n=1}^{\infty}$ *is a countable collection of disjoint subsets of* $\mathbb{R}$ *that covers* $E \subseteq \mathbb{R}$*, then*

$$\mu E = \sum_{n=1}^{\infty} \mu E_n.$$

Consequently, the Lebesgue outer measure m^* *is* ***not*** σ*-additive.*

Proof

Suppose to the contrary that such a μ exists.

Step 1 Consider the relation $\sim$ on $\mathbb{R}$ such that $x \sim y$ if $x - y \in \mathbb{Q}$.

Claim: $\sim$ is an equivalence relation

- (**reflexivity**) We know that $0 \in \mathbb{Q}$ and $x - x = 0$. Thus $x \sim x$.
- (**symmetry**) Since $\mathbb{Q}$ is a **field**, it is closed under multiplication, and $-1 \in \mathbb{Q}$. Thus if $x \sim y$, then $x - y \in \mathbb{Q}$, and so $(-1)(x-y) = y - x \in \mathbb{Q}$, which means $y \sim x$.
- (**transitivity**) Again, since $\mathbb{Q}$ is a field, it is closed under (this time) addition. Thus

$$\begin{aligned} x \sim y \wedge y \sim z &\implies (x-y), (y-z) \in \mathbb{Q} \\ &\implies (x-y) + (y-z) = x - z \in \mathbb{Q}. \end{aligned}$$

Thus $x \sim z$.

This proves the claim. $\dashv$

Let

$$[x] := x + \mathbb{Q} := \{x + q : q \in \mathbb{Q}\}$$

denote the equivalence class of x wrt $\sim$. Note that the set of equivalence classes, which we shall represent as

$$\mathcal{F} := \{[x] : x \in \mathbb{R}\},$$

partitions $\mathbb{R}$, i.e.

- $[x] = [y] \iff x - y \in \mathbb{Q}$; and
- $[x] \cap [y] = \emptyset$ otherwise.

Note that since $\mathbb{Q}$ is **dense** in $\mathbb{R}$, we have that $[x] = x + \mathbb{Q}$ is also dense in $\mathbb{R}$, for all $x \in \mathbb{R}$. Then for each [2] $F \in \mathcal{F}$, $\exists x_F \in F$ such that

$$0 \leq x_F \leq 1.$$

Now consider the set

$$\mathbb{V} := \{x_F : F \in \mathcal{F}\} \subseteq [0, 1],$$

which is called **Vitali's Set**.

[2] Notice that here, we have invoked the Axiom of Choice.

Step 2 Since $\mathcal{F}$ partitions $\mathbb{R}$, we have that

$$\begin{aligned} \mathbb{R} = \dot{\bigcup_{F \in \mathcal{F}}} F = \dot{\bigcup_{F \in \mathcal{F}}} [x_F] \\ = \dot{\bigcup_{F \in \mathcal{F}}} x_F + \mathbb{Q} \\ = \mathbb{V} + \mathbb{Q} := \{x + q : q \in \mathbb{Q}, x \in \mathbb{V}\}. \end{aligned}$$

Step 3 Claim: $p \neq q \in \mathbb{Q} \implies (\mathbb{V} + p) \cap (\mathbb{V} + q) = \emptyset$ Suppose not, and suppose $\exists y \in (\mathbb{V} + p) \cap (\mathbb{V} + q)$. Then $\exists F_1, F_2 \in \mathcal{F}$ such that

$$y = x_{F_1} + p = x_{F_2} + q. \tag{3.1}$$

Then we may rearrange the above equation to get

$$x_{F_1} - x_{F_2} = q - p \in \mathbb{Q}.$$

This implies that

$$[x_{F_1}] = [x_{F_2}] \implies F_1 = F_2$$

since $\mathbb{V}$ consists of one unique representative from each of the equivalence classes. However, this would mean that

$$x_{F_1} = x_{F_2}.$$

Since $p \neq q$, we have that

$$x_{F_1} + p \neq x_{F_2} + q,$$

which contradicts Equation (3.1). Thus

$$(\mathbb{V} + p) \cap (\mathbb{V} + q) = \emptyset,$$

as claimed. $\dashv$

This in turn means that the $\mathbb{V} + q$, for each $q \in \mathbb{Q}$, also partitions $\mathbb{R}$. In other words, if we write $\mathbb{Q} = \{p_n\}_{n=1}^{\infty}$, then

$$\mathbb{R} = \mathbb{V} + \mathbb{Q} = \dot{\bigcup_{n=1}^{\infty}} \mathbb{V} + p_n.$$

Now, note that

$$0 \neq \mu\mathbb{R} \overset{(1)}{=} \sum_{n=1}^{\infty} \mu(\mathbb{V} + p_n) \overset{(2)}{=} \sum_{n=1}^{\infty} \mu(\mathbb{V}),$$

where (1) is by μ being σ-additive and (2) is by μ being translation invariant, both directly from our assumptions. This means that

$$\mu\mathbb{V} > 0.$$

Step 4 Now consider $S = \mathbb{Q} \cap [0, 1]$ such that S is denumerable. Write

$$S = \{s_n\}_{n=1}^{\infty}.$$

Note that for all $n \geq 1$,

$$\mathbb{V} \subseteq [0,1] \implies \mathbb{V} + s_n \subseteq [0,2],$$

and as proven above

$$i \neq j \implies (\mathbb{V} + s_i) \cap (\mathbb{V} + s_j) = \emptyset.$$

Thus it follows that

$$\mu\left(\dot{\bigcup_{n=1}^{\infty}} \mathbb{V} + s_n \right) = \sum_{n=1}^{\infty} \mu(\mathbb{V} + s_n) = \sum_{n=1}^{\infty} \mu(\mathbb{V}) = \infty.$$

Also,

$$\begin{aligned} \mu\left(\dot{\bigcup_{n=1}^{\infty}} \mathbb{V} + s_n \right) &= \sum_{n=1}^{\infty} \mu(\mathbb{V} + s_n) \\ &\leq \mu([0,2]) = \mu([0,1] \cup ([0,1] + 1)) \\ &\leq \mu[0,1] + \mu([0,1] + 1) \\ &= 2\mu([0,1]) = 2 < \infty, \end{aligned}$$

contradicting what we have right above.

Therefore, no such μ exists. □

With the realization of ☕Theorem 12, we find ourselves facing a losing dilemma: we may either

1. be happy with the Lebesgue outer measure m^* for all subsets $E \subseteq \mathbb{R}$, which would agree with our intuitive notion of length, at the price of σ-additivity; or

2. restrict the domain of our function m^* to some family of subsets of $\mathbb{R}$, where m^* would have σ-additivity.

We shall adopt the second approach. We shall call the collection of sets where m^* has σ-additivity as the collection of **Lebesgue measurable sets**.

3.2 Lebesgue Measure

We shall first introduce **Carathéodory's** definition of a Lebesgue measurable set.

Definition 16 (Lebesgue Measureable Set)

A set $E \subseteq \mathbb{R}$ is said to be ***Lebesgue measurable*** *if, $\forall X \subseteq \mathbb{R}$,*

$$m^* X = m^*(X \cap E) + m^*(X \setminus E).$$

We denote the collection of all Lebesgue measurable sets as $\mathfrak{M}(\mathbb{R})$.

Remark 3.2.1

Since we shall almost exclusively focus on the Lebesgue measure, we shall hereafter refer to "Lebesgue measurable sets" as simply "measurable sets".

Note 3.2.1

I shall quote and paraphrase this remark from our course notes [3]:

> *Informally, we see that a set $E \subseteq \mathbb{R}$ is measurable provided that it is a "universal slicer", that it "slices" every other set X into two* ***disjoint*** *sets, into where the Lebesgue outer measure is* ***σ-additive****.*

Also, note that we get the following inequality for free, simply from σ-subadditivity of m^*:

$$m^* X \leq m^*(X \cap E) + m^*(X \setminus E).$$

Thus, it suffices for us to check if the reverse inequality holds for all sets $X \subseteq \mathbb{R}$.

[3] Marcoux, L. W. (2019). *PMath 450 Introduction to Lebesgue Measure and Fourier Analysis*. (n.p.)

Before ploughing forward to getting out hands dirty with examples, let us first study a result on a structure of $\mathfrak{M}(\mathbb{R})$ that is rather interesting. [4]

[4] For those who has dirtied themselves in the world of probability and statistics, especially probability theory, get ready to get excited!

Definition 17 (Algebra of Sets)

A collection $\Omega \subseteq \mathcal{P}(\mathbb{R})$ is said to be an ***algebra of sets*** *if*

1. $\mathbb{R} \in \Omega$;
2. *(**closed under complementation**)* $E \in \Omega \implies E^C \in \Omega$; *and*
3. *(**closed under finite union**) given* $N \geq 1$ *and* $\{E_n\}_{n=1}^N \subseteq \Omega$, *then*

$$\bigcup_{n=1}^{N} E_n \in \Omega.$$

We say that Ω *is a* ***σ-algebra of sets*** *if*

1. Ω *is an algebra of sets; and*
2. *(**closed under countable union**) if* $\{E_n\}_{n=1}^\infty \subseteq \Omega$, *then*

$$\bigcup_{n=1}^{\infty} E_n \in \Omega.$$

Note 3.2.2

We often call a σ-algebra of sets as simply a σ-algebra.

Theorem 13 ($\mathfrak{M}(\mathbb{R})$ is a σ-algebra)

The collection $\mathfrak{M}(\mathbb{R})$ of Lebesgue measurable sets in $\mathbb{R}$ is a σ-algebra.

Due to time constraints, we shall prove the first 2 requirements in this lecture and prove the last requirement next time (which is also really long).

Proof

$\mathbb{R} \in \mathfrak{M}(\mathbb{R})$ Observe that $\forall X \subseteq \mathbb{R}$,

$$m^* X = m^* X + 0 = m^* X + m^* \emptyset = m^*(X \cap \mathbb{R}) + m^*(X \setminus \mathbb{R}) \qquad \square$$

$E \in \mathfrak{M}(\mathbb{R}) \implies E^C \in \mathfrak{M}(\mathbb{R})$ Observe that $\forall X \subseteq \mathbb{R}$, since $E \in \mathfrak{M}(\mathbb{R})$, we have

$$\begin{aligned} m^* X &= m^*(X \cap E) + m^*(X \setminus E) \\ &= m^*(X \cap (E^C)^C) + m^*(X \cap E^C) && A \setminus B = A \cap B^C \\ &= m^*(X \setminus E^C) + m^*(X \cap E^C) \\ &= m^*(X \cap E^C) + m^*(X \setminus E^C) && \textit{rearrangement} \end{aligned}$$

Thus $E^C \in \mathfrak{M}(\mathbb{R})$.

4 Lecture 4 May 16th 2019

4.1 Lebesgue Measure (Continued)

Recalling the last theorem we were in the middle of proving, it remains for us to prove that $\mathfrak{M}(\mathbb{R})$ is closed under arbitrary unions of its elements.

But before we dive in, let's first have a little pep talk.

Strategy

Since m^* is σ-subadditive, given $\{E_n\}_{n=1}^{\infty}$, we need only prove that $\forall X \subseteq \mathbb{R}$,

$$m^* X \geq m^* \left(X \cap \bigcup_{n=1}^{\infty} E_n \right) + m^* \left(X \setminus \bigcup_{n=1}^{\infty} E_n \right).$$

Recall our discussion near the end of Section 3.1. We want σ-additivity, especially when we are given a set of disjoint intervals. However, our E_n's are arbitrary, and so they are not necessarily disjoint.

It helps if one has seen how we can slice $\mathbb{R}$ up into disjoint unions, and consequently we can do so for any of its subsets. We shall not take that for granted and immediately use it, but we shall work through this proof in the spirit of that. We shall see how we can slice $\mathbb{R}$ up in A1.

Once we can, in some way, express $\bigcup_{n=1}^{\infty} E_n$ as a disjoint union of intervals, we will then show that, indeed, we have σ-additivity instead of σ-subadditivity on this disjoint union.

Proof

$\mathfrak{M}(\mathbb{R})$ is closed under arbitrary unions Suppose $\{E_n\}_{n=1}^{\infty} \subseteq \mathfrak{M}(\mathbb{R})$. To show that $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{M}(\mathbb{R})$, WTS

$$m^* X = m^* \left(X \cap \bigcup_{n=1}^{\infty} E_n \right) + m^* \left(X \setminus \bigcup_{n=1}^{\infty} E_n \right).$$

Since m^* is σ-subadditive, it suffices for us to show that

$$m^* X \geq m^* \left(X \cap \bigcup_{n=1}^{\infty} E_n \right) + m^* \left(X \setminus \bigcup_{n=1}^{\infty} E_n \right). \tag{4.1}$$

Step 1 Consider

$$H_n = \bigcup_{i=1}^{n} E_i, \quad \forall n \geq 1.$$

Claim: $H_n \in \mathfrak{M}(\mathbb{R}), \forall n \geq 1$ We shall prove this by induction on n.

When $n = 1$, we have $H_1 = E_1 \in \mathfrak{M}(\mathbb{R})$ by assumption, and so we are done. Suppose that $H_k \in \mathfrak{M}(\mathbb{R})$ for some $k \in \mathbb{N}$. Consider $n = k + 1$.

Since we will need the piece $X \cap H_{k+1}$, first, notice that

$$X \cap H_{k+1} = X \cap (H_k \cup E_{k+1}) = (X \cap H_k) \cup ((X \setminus H_k) \cap E_{k+1}),$$

and in particular that

$$X \cap H_{k+1} = X \cap (H_k \cup E_{k+1}) \subseteq (X \cap H_k) \cup ((X \setminus H_k) \cap E_{k+1}). \tag{4.2}$$

This may be (will be) useful later on, and we can guess that we will be using σ-subadditivity on this.

By the IH, since $H_k \in \mathfrak{M}(\mathbb{R})$, we have

$$m^* X = m^*(X \cap H_k) + m^*(X \setminus H_k).$$

Notice the similarity between the above equation and Equation (4.2), where we are just off by that $\cap E_{k+1}$.

Since $E_{k+1} \in \mathfrak{M}(\mathbb{R})$, we have

$$m^*(X \setminus H_k) = m^*(X \setminus H_k \cap E_{k+1}) + m^*(X \setminus H_k \setminus E_{k+1}).$$

To clean the above equation up a little bit, notice that by **De Mor-**

gan's Law,

$$X \setminus H_k \setminus E_{k+1} = X \cap \bigcup_{i=1}^{k} E_i^C \cap E_{k+1}^C = X \setminus H_{k+1}.$$

So

$$m^*(X \setminus H_k) = m^*(X \setminus H_k \cap E_{k+1}) + m^*(X \setminus H_{k+1}).$$

Thus

$$m^* X = m^*(X \cap H_k) + m^*(X \setminus H_k \cap E_{k+1}) + m^*(X \setminus H_{k+1}).$$

Using Equation (4.2) and σ-subadditivity, we have that

$$m^* X \geq m^*(X \cap H_{k+1}) + m^*(X \setminus H_{k+1}),$$

which is what we need. Thus $\forall k \geq 1$, $H_k \in \mathfrak{M}(\mathbb{R})$. $\dashv$

Step 2 Consider $F_1 = H_1 = E_1 \in \mathfrak{M}(\mathbb{R})$, and for $k \geq 2$,

$$F_k = H_k \setminus H_{k-1} = H_k \cap H_{k-1}^C.$$

[1] Claim: $\forall k \geq 2$, $F_k \in \mathfrak{M}(\mathbb{R})$ First, notice that

$$F_k^C = \left(H_k \cap H_{k+1}^C \right)^C = H_k^C \cup H_{k+1}.$$

By **step 1** [2], we have that $F_k^C \in \mathfrak{M}(\mathbb{R})$, and thus by closure under complementation, $F_k \in \mathfrak{M}(\mathbb{R})$.

[1] Note that we cannot assume that $\mathfrak{M}(\mathbb{R})$ is closed under finite intersections because that is part of what we want to prove.

[2] **I need to get this clarified.**

Also, note that the F_i's are pairwise disjoint. Suppose not, i.e. that $\exists x \in F_a \cap F_b$ for some $a, b \geq 1$ and $a \neq b$. Wlog, wma $a < b$. Note that $H_a \subseteq H_b$, since

$$H_a = \bigcup_{i=1}^{a} E_i \subsetneq \bigcup_{i=1}^{b} E_i = H_b.$$

Since $F_b = H_b \setminus H_{b-1}$,

$$x \in F_b \implies x \notin \bigcup_{i=1}^{b-1} E_i \supseteq \bigcup_{i=1}^{a} E_i,$$

and so $x \notin E_i$ for $1 \leq i \leq a \leq b-1$. But we assumed that

$$x \in F_a = H_a \setminus H_{a-1},$$

i.e. it must be that $x \in E_a$, a contradiction.

Step 3 We now have

$$E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} H_i = \dot{\bigcup_{i=1}^{\infty}} F_i.$$

Equation (4.1) becomes [3]

$$m^* X \geq m^* \left(X \cap \left(\dot{\bigcup_{i=1}^{\infty}} F_i \right) \right) + m^* \left(X \setminus E \right).$$

[3] I refrained from changing the second term to the disjoint union. Retrospectively (i.e. once you're done with the proof), it makes sense to not consider this move, since there is no point looking at X take away a bunch of disjoint intervals.

Since the F_i's are disjoint, we expect

$$m^* \left(X \cap \dot{\bigcup_{i=1}^{\infty}} F_i \right) = \sum_{i=1}^{\infty} m^*(X \cap F_i).$$

i.e. for every n,

$$m^* \left(X \cap \dot{\bigcup_{i=1}^{n}} F_i \right) = \sum_{i=1}^{n} m^*(X \cap F_i).$$

Let's prove this inductively. It is clear that case $n = 1$ is trivially true. Suppose that this is true up to some $k \in \mathbb{N}$. Consider case $n = k+1$. Since $F_{k+1} \in \mathfrak{M}(\mathbb{R})$, we have that [4]

[4] This is quite a smart trick!

$$\begin{aligned}
& m^* \left(X \cap \bigcup_{i=1}^{k+1} F_i \right) \\
&= m^* \left(X \cap \bigcup_{i=1}^{k+1} F_i \cap F_{k+1} \right) + m^* \left(\left(X \setminus \bigcup_{i=1}^{k=1} F_i \right) \setminus F_{k+1} \right) \\
&= m^*(X \cap F_{k+1}) + m^* \left(X \cap \bigcup_{i=1}^{k} F_i \right) \\
&= m^*(X \cap F_{k+1}) + \textstyle\sum_{i=1}^{k} m^*(X \cap F_i) \qquad \text{IH} \\
&= \sum_{i=1}^{k+1} m^*(X \cap F_i).
\end{aligned}$$

Our claim is complete by induction.

Step 4 With **Step 3**, Equation (4.1) has become

$$m^* X \geq \sum_{i=1}^{\infty} m^*(X \cap F_i) + m^*(X \setminus E).$$

[5] Since $H_k \in \mathfrak{M}(\mathbb{R})$ for each $k \geq 1$, we have

[5] This is a reward for the clear-minded, cause I certainly did not find it an obvious step to take.

$$m^* X = m^*(X \cap H_k) + m^*(X \setminus H_k). \tag{$*$}$$

Since

$$H_k = \bigcup_{i=1}^{k} E_i = \bigcup_{i=1}^{\infty} E_i = E,$$

we have that

$$X \setminus H_k \supseteq X \setminus E,$$

for each $k \geq 1$. Thus by monotonicity, Equation ($*$) becomes

$$\begin{aligned} m^* X &\geq m^*(X \cap H_k) + m^*(X \setminus E) \\ &= m^* \left(X \cap \left(\bigcup_{i=1}^{\infty} F_i \right) \right) + m^*(X \setminus E) \\ &= \sum_{i=1}^{k} m^*(X \cap F_i) + m^*(X \setminus E), \end{aligned}$$

for each $k \geq 1$.

By letting $k \to \infty$, we have that

$$m^* X \geq \sum_{i=1}^{\infty} m^*(X \cap F_i) + m^*(X \setminus E).$$

Note that

$$X \cap E = X \cap \bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} (X \cap F_i).$$

By σ-subadditivity, we have that

$$m^*(X \cap E) \leq \sum_{i=1}^{\infty} m^*(X \cap F_i).$$

Therefore

$$m^* X \geq m^*(X \cap E) + m^*(X \setminus E),$$

which is what we want! □

Note 4.1.1 (Post-mortem for proof of Theorem 13)

*In steps 1 - 3, we try to slice $\bigcup_{n=1}^{\infty} E_n$ into disjoint **measurable** intervals F_i's. Along the process of constructing them, it is the showing of them being measurable that takes up most of the proof, since we require induction.*

Proposition 14 (Some Lebesgue Measurable Sets)

1. *If $E \subseteq \mathbb{R}$ and $m^* E = 0$, then E is Lebesgue measurable.*
2. $\forall b \in \mathbb{R}$, $(-\infty, b) \in \mathfrak{M}(\mathbb{R})$.
3. *Every open and every closed set is Lebesgue measurable.*

Proof

1. Let $X \subseteq \mathbb{R}$. Note that $X \setminus E \subseteq X$, and so σ-subadditivity gives

$$m^* X \geq m^*(X \setminus E). \tag{4.3}$$

On the other hand, $X \cap E \subseteq E$, and so

$$m^*(X \cap E) \leq m^* E = 0 \implies m^*(X \cap E) = 0.$$

Thus, from Equation (4.3),

$$m^* X \geq ml * (X \setminus E) = m^*(X \cap E) + m^*(X \setminus E).$$

Hence $E \in \mathfrak{M}(\mathbb{R})$ as required.

2. Let $b \in \mathbb{R}$ and $X \subseteq \mathbb{R}$ be arbitrary. WTS

$$m^* X \geq m^*(X \cap (-\infty, b)) + m^*(X \setminus (-\infty, b)).$$

[6] Let $E = (-\infty, b)$. Note that if $m^* X = \infty$, then there is nothing to show. Thus WMA $m^* X < \infty$. In this case, let $\varepsilon > 0$, and $\{I_n\}_{n=1}^{\infty}$ a

[6] We will look at $X \cap (\infty, b)$ and $X \setminus (-\infty, b)$ more closely, and then realize that since we can cover X, we can "extend" this cover for these disjoint pieces by taking intersections and set removals on each of the covering sets.

cover of X by open intervals, where we write

$$I_n = (a_n, b_n)$$

for each $n \geq 1$, so that [7]

$$\sum_{n=1}^{\infty} \ell(I_n) < m^* X + \varepsilon.$$

[7] Note that this is legitimate because $m^* X$ is the infimum of such sums on the LHS, and we can definitely find such a cover as a result. Also, there is no harm in assuming that each of the I_n's are non-empty, since we may simply remove all the empty I_n's from the cover.

For each $n \geq 1$, consider the sets

$$J_n = I_n \cap E + I_n \cap (-\infty, b)$$

and

$$K_n = I_n \setminus E = I_n \setminus (\infty, b) = I_n \cap [b, \infty).$$

The following table captures all possible J_n's and K_n's:

Case	1	2	3
b	$> b_n$	$\in I_n$	$< a_n$
J_n	I_n	(a_n, b)	$\emptyset$
K_n	$\emptyset$	$[b, b_n)$	I_n

Table 4.1: Possible outcomes of J_n and K_n, for each $n \geq 1$

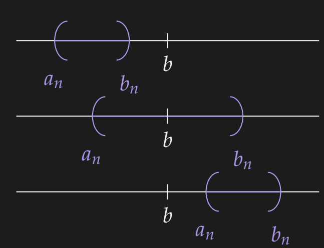

Figure 4.1: Three possible scenarios of where b stands for different I_n's

Notice that $\{J_n\}_{n=1}^{\infty}$ is an open cover for $X \cap E$. $\{K_n\}_{n=1}^{\infty}$ is also a cover of $X \setminus E$ but it is not an open cover (the only covers of which we consider in this course). Thus, we consider a small extension L_n of K_n such that

- if $K_n = \emptyset$, then $L_n = \emptyset$;
- if $K_n = I_n$, then $L_n = I_n$; and
- if $K_n = [b, b_n)$, then $L_n = \left(b - \frac{\varepsilon}{2^n}, b_n\right)$.

Then $\{L_n\}_{n=1}^{\infty}$ is a cover of $X \setminus E$. By σ-subadditivity of m^*, we have that

$$m^*(X \cap E) \leq \sum_{n=1}^{\infty} \ell(J_n)$$

and

$$m^*(X \setminus E) \leq \sum_{n=1}^{\infty} \ell(L_n).$$

Thus

$$m^*(X \cap E) + m^*(X \setminus E) \leq \sum_{n=1}^{\infty} \left(\ell(J_n) + \ell(L_n)\right).$$

Now, notice that in cases 1 and 3,

$$\ell(J_n) + \ell(L_n) = \ell(I_n).$$

In case 2, we have that

$$(\ell(J_n) + \ell(L_n)) - \ell(I_n) < \frac{\varepsilon}{2^n}$$

and so

$$\ell(J_n) + \ell(L_n) < \ell(I_n) + \frac{\varepsilon}{2^n}.$$

Therefore

$$\begin{aligned}
&m^*(X \cap E) + m^*(X \setminus E) \\
&\leq \sum_{n=1}^{\infty} (\ell(J_n) + \ell(L_n)) \\
&\leq \sum_{n=1}^{\infty} \left(\ell(I_n) + \frac{\varepsilon}{2^n} \right) \\
&= \sum_{n=1}^{\infty} \ell(I_n) + \varepsilon \\
&< (m^* X + \varepsilon) + \varepsilon \\
&= m^* X + 2\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have that

$$m^* X \geq m^*(X \cap E) + m^*(X \setminus E),$$

and since X is arbitrary, we have that $E = (-\infty, b) \in \mathfrak{M}(\mathbb{R})$.

3. Wlog, suppose $a < b \in \mathbb{R}$. By part 2, we have that

$$(-\infty, b) \in \mathfrak{M}(\mathbb{R}),$$

and similarly, for $n \geq 1$,

$$\left(\infty, a + \frac{1}{n} \right) \in \mathfrak{M}(\mathbb{R}).$$

Since $\mathfrak{M}(\mathbb{R})$ is a σ-algebra, we have that

$$\left[a + \frac{1}{n}, \infty \right) = \left(-\infty, a + \frac{1}{n} \right)^C \in \mathfrak{M}(\mathbb{R}),$$

for each $n \geq 1$. Consequently,

$$(a, \infty) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, \infty \right) \in \mathfrak{M}(\mathbb{R}).$$

Therefore, we have that

$$(a, b) = (-\infty, b) \cap (a, \infty) \in \mathfrak{M}(\mathbb{R}).$$

[8] Since every open set $G \subseteq \mathbb{R}$ is a countable disjoint union of open intervals in $\mathbb{R}$, it follows that $G \in \mathfrak{M}(\mathbb{R})$ since $\mathfrak{M}(\mathbb{R})$ is a σ-algebra. If $F \subseteq \mathbb{R}$ is closed, notice that

$$F^C = G \in \mathfrak{M}(\mathbb{R})$$

since G is open, and so by closure under complementation of σ-algebras, $F \in \mathfrak{M}(\mathbb{R})$. □

[8] We shall prove this in A1.

Definition 18 (Lebesgue Measure)

Let m^* denote the Lebesgue outer measure on $\mathbb{R}$. We define the ***Lebesgue measure*** *m to be*

$$m = m^* \restriction_{\mathfrak{M}(\mathbb{R})},$$

i.e. $\forall E \in \mathfrak{M}(\mathbb{R})$, we have that

$$mE = m^*E = \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) \mid E \subseteq \bigcup_{n=1}^{\infty} I_n \right\}.$$

In A2, we shall prove that

Theorem 15 (σ-additivity of the Lebesgue Measure on Lebesgue Measurable Sets)

The Lebesgue measure is σ-additive on $\mathfrak{M}(\mathbb{R})$, i.e. if $\{E_n\}_{n=1}^{\infty} \subseteq \mathfrak{M}(\mathbb{R})$ with $E_i \cap E_j = \emptyset$ for all $i \neq j$, then

$$m \bigcup_{n=1}^{\infty} E_n = \sum_{n=1}^{\infty} mE_n.$$

Corollary 16 (Existence of Non-Measurable Sets)

There exists non-measurable sets.

Proof

Suppose not, i.e. $\mathfrak{M}(\mathbb{R}) = \mathcal{P}(\mathbb{R})$. Then $m = m^*$ is a translation invariant outer measure on $\mathbb{R}$, with $m^*\mathbb{R} = \infty > 0$, $m^*[0,1] = 1 < \infty$, and m^* is σ-additive, which contradicts Theorem 12. Thus $\mathfrak{M}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$. □

The following proposition is left as an exercise.

Proposition 17 (Non-measurability of the Vitali Set)

The Vitali set $\mathbb{V}$, defined in Theorem 12, is not measurable.

Exercise 4.1.1
Prove Proposition 17.

Definition 19 (σ-algebra of Borel Sets)

The σ-algebra of sets generated by the collection

$$\mathfrak{G} := \{ G \subseteq \mathbb{R} : G \text{ is open } \}$$

is called the ***σ-algebra of Borel sets*** *of $\mathbb{R}$, and is denoted by*

$$\mathfrak{Bor}(\mathbb{R}).$$

Note 4.1.2

Since $\mathfrak{Bor}(\mathbb{R})$ is generated by open sets in $\mathbb{R}$ and all open subsets of $\mathbb{R}$ are Lebesgue measurable (cf. Proposition 14), we have that

$$\mathfrak{Bor}(\mathbb{R}) \subseteq \mathfrak{M}(\mathbb{R}).$$

Remark 4.1.1

Since $\mathfrak{Bor}(\mathbb{R})$ is a σ-algebra, and it is, in particular, generated by open subsets of $\mathbb{R}$, it also contains all of the closed subsets of $\mathbb{R}$. Thus, we could have instead defined $\mathfrak{Bor}(\mathbb{R})$ to be the σ-algebra of subsets of $\mathbb{R}$ generated by the collection

$$\mathfrak{F} := \{F \subseteq \mathbb{R} : F \text{ is closed }\},$$

and in turn conclude that $\mathfrak{Bor}(\mathbb{R})$ contains $\mathfrak{G}$.

Remark 4.1.2

Let $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$, with $\emptyset, \mathbb{R} \in \mathcal{A}$. Let

$$\mathcal{A}_\sigma := \left\{ \bigcup_{n=1}^{\infty} A_n : A_n \in \mathcal{A}, n \geq 1 \right\}$$
$$\mathcal{A}_\delta := \left\{ \bigcap_{n=1}^{\infty} A_n : A_n \in \mathcal{A}, n \geq 1 \right\}.$$

*We call the elements of $\mathcal{A}_\sigma$ as **$\mathcal{A}$-sigma sets**, and elements of $\mathcal{A}_\delta$ as **$\mathcal{A}$-delta sets**.*

Recalling our definitions

$$\mathfrak{G} = \{G \subseteq \mathbb{R} \mid G \text{ is open }\}$$
$$\mathfrak{F} = \{F \subseteq \mathbb{R} \mid F \text{ is closed }\}$$

from above, notice that

$$\mathfrak{G}_\delta = \left\{ \bigcap_{n=1}^{\infty} G_n \mid G_n \in \mathfrak{G}, n \geq 1 \right\},$$

*which is a **countable intersection of open subsets** of $\mathbb{R}$, and*

$$\mathfrak{F}_\sigma = \left\{ \bigcup_{n=1}^{\infty} F_n \mid F_n \in \mathfrak{F}, n \geq 1 \right\},$$

*which is a **countable union of closed subsets** of $\mathbb{R}$, are both subsets of $\mathfrak{Bor}(\mathbb{R})$.*

As mentioned before, the definition of which we provided for a

Lebesgue measurable set is from **Carathéodory**, which is not the most intuitive definition. We shall now show that it is equivalent to the original definition of which Lebesgue himself has provided.

Theorem 18 (Carathéodory's and Lebesgue's Definition of Measurability)

Let $E \subseteq \mathbb{R}$. TFAE:

1. *E is Lebesgue measurable (Carathéodory).*
2. *$\forall \varepsilon > 0$, there exists an open $G \supseteq E$ such that*
$$m^*(G \setminus E) < \varepsilon.$$
3. *There exists a $\mathfrak{G}_\delta$-set H such that $E \subseteq H$ and*
$$m^*(H \setminus E) = 0.$$

Proof

(1) $\implies$ (2) If we can find such a G that is open, then since E is Lebesgue measurable, we have

$$mG = m(G \cap E) + m(G \setminus E) = mE + m(G \setminus E),$$

and so

$$m(G \setminus E) = mG - mE. \tag{4.4}$$

So if we can construct such a G, that is particularly small enough (within ε-bigger) to contain E, our statement is good as done.

Case 1: $mE < \infty$ In this case, we may consider a cover $\{I_n\}_{n=1}^{\infty}$ of E such that

$$\sum_{n=1}^{\infty} \ell(I_n) < mE + \varepsilon.$$

Then we may simply let $G = \bigcup_{n=1}^{\infty} I_n$. Note that since $\mathfrak{M}(\mathbb{R})$ is a

σ-algebra, $G \in \mathfrak{M}(\mathbb{R})$. Thus by monotonicity,

$$mG = m\left(\bigcup_{n=1}^{\infty} I_n\right) \leq \sum_{n=1}^{\infty} mI_n = \sum_{n=1}^{\infty} \ell(I_n) < mE + \varepsilon.$$

With this, Equation (4.4) becomes

$$m(G \setminus E) < mE + \varepsilon - mE = \varepsilon.$$

Case 2: $\forall r \in \mathbb{R},\ mE > r$ Consider

$$E_k = [-k, k] \cap E$$

[9]for each $k \geq 1$. By Proposition 14, closed sets are Lebesgue measurable, and so for each $k \geq 1$, $E_k \in \mathfrak{M}(\mathbb{R})$. Note that

$$E = \bigcup_{k \geq 1} E_k.$$

[10] Note that $E_k \subseteq [-k, k]$, and so

$$mE_k \leq m[-k, k] = 2k < \infty.$$

Using a similar approach as in **Case 1**, we can construct an open set G_k such that $G_k \supseteq E_k$, and

$$m(G_k \setminus E_k) < \frac{\varepsilon}{2^k},$$

for each $k \geq 1$. Now let

$$G := \bigcup_{k \geq 1} G_k \supseteq \bigcup_{k \geq 1} E_k = E.$$

Note that if $x \in G \setminus E$, then $x \notin E_k$ for all $k \geq 1$, and $\exists N \geq 1$ such that $x \in G_N$. In particular, we have that

$$x \in G_N \setminus E_N,$$

and so

$$G \setminus E \subseteq \bigcup_{k \geq 1} G_k \setminus E_k$$

[11]. Therefore

[9] **I should get clarification for my understanding of this approach.** We picked closed intervals instead of open ones so that we deal with the possible quirkiness of E.

[10] It would be a quick job if we take the union of the E_k's but note that the E_k's are not necessarily open!

[11] It is, however, true that equality holds, and it is not difficult to prove so.

$$m(G \setminus E) \leq \sum_{k \geq 1} m(G_k \setminus E_k) \leq \sum_{k \geq 1} \frac{\varepsilon}{2^k} = \varepsilon.$$

(2) $\implies$ (3) By (2), for each $n \geq 1$, let $G_n \supseteq E$ such that

$$m(G_n \setminus E) < \frac{1}{n}.$$

Let $H := \bigcap_{n \geq 1} G_n$, which then $H \in \mathfrak{G}_\delta$. Also, since $E \subseteq G_n$ for all $n \geq 1$, we have $E \subseteq H$. Also, $H \subseteq G_n$ for each n. Thus

$$H \setminus E \subseteq G_n \setminus E,$$

for each $n \geq 1$. By monotonicity,

$$m(H \setminus E) \leq m(G_n \setminus E) < \frac{1}{n}$$

for each $n \geq 1$. Therefore

$$m(H \setminus E) = 0.$$

(3) $\implies$ (1) Notice that $\mathfrak{G}_\delta \subseteq \mathfrak{Bor}(\mathbb{R}) \subseteq \mathfrak{M}(\mathbb{R})$. Suppose $G \in \mathfrak{G}_\delta$, and $E \subseteq H$ such that

$$m(H \setminus E) = 0.$$

By Proposition 14, $H \setminus E \in \mathfrak{M}(\mathbb{R})$. Since $\mathfrak{M}(\mathbb{R})$ is a σ-algebra, notice that

$$E = H \setminus (H \setminus E) = H \cap (H \cap E^C)^C = H \cap H^C \cup E \in \mathfrak{M}(\mathbb{R}). \qquad \square$$

5 Lecture 5 May 21st 2019

5.1 Lebesgue Measure (Continued 2)

Recall from Corollary 6 that any countable subset $E \subseteq \mathbb{R}$ has zero Lebesgue outer measure. From Proposition 14, we have that $E \in \mathfrak{M}(\mathbb{R})$ and so $mE = m^*E = 0$. This shows that every countable set is Lebesgue measurable with Lebesgue measure zero.

> *But is the converse true? I.e., is every Lebesgue measurable set with Lebesgue measure zero countable?*

We shall show that this is not true by giving a counterexample. We shall now construct an **uncountable set** C that has measure zero.

Example 5.1.1 (The Cantor Set)

Let $C_0 = [0, 1]$. Note that C_0 is compact and

$$m^* C_0 = 1 < \infty.$$

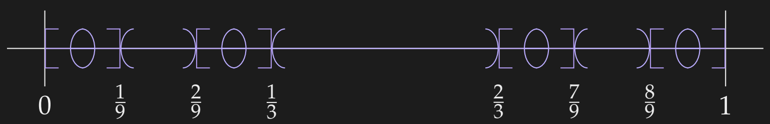

Figure 5.1: Cantor set showing up to $n = 2$, with the excluded interval in $n = 3$ shown.

Let

$$C_1 = C_0 \setminus \left(\frac{1}{3}, \frac{2}{3} \right).$$

Then C_1 is closed [1] and $C_0 \supseteq C_1$.

[1] C_1 is an intersection of 2 closed sets.

Let

$$C_2 = C_1 \setminus \left(\left(\frac{1}{9}, \frac{2}{9} \right) \cup \left(\frac{7}{9}, \frac{8}{9} \right) \right).$$

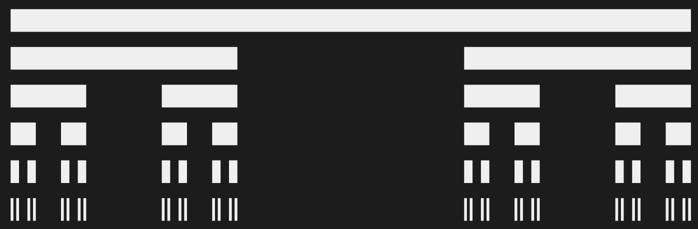

Figure 5.2: An illustration of the Cantor Set from `https://mathforum.org/mathimages/index.php/Cantor_Set`.

Then C_2 is closed and $C_1 \supseteq C_2$.

We continue this process indefinitely, and construct C_n for each $n \geq 1$, where

$$C_n = \frac{1}{3}C_{n-1} \cup \left(\frac{2}{3} + \frac{1}{3}C_{n-1} \right).$$

Then C_n will consist of 2^n disjoint closed intervals. Thus each C_n is compact and measurable. Moreover,

$$m(C_n) = \left(\frac{2}{3} \right)^n,$$

for each $n \geq 1$.

Also, we have that

$$C_0 \supseteq C_1 \supseteq C_2 \supseteq \ldots$$

is a **descending chain of measurable sets**. Note that the sequence $\{C_n\}_{n=0}^{\infty}$ has the **finite intersection property**, and since $\mathbb{R}$ is compact, the set

$$C := \bigcap_{n=1}^{\infty} C_n,$$

which we shall call it the **Cantor Set**, is non-empty [2].

[2] See FIP and Compactness from PMATH 351

Now from A2, we have that

$$mC = \lim_{n \to \infty} mC_n = \lim_{n \to \infty} \left(\frac{2}{3} \right)^n = 0.$$

We shall now show that C is uncountable. To do this, we shall use the **ternary representation** for each $x \in [0, 1]$. In particular, for each $x \in [0, 1]$, we write

$$x = 0.x_1 x_2 x_3 \ldots,$$

where each $x_i \in \{0, 1, 2\}$ for all $i \geq 1$. Note that in base 10, we can express

$$x = \sum_{k=1}^{\infty} \frac{x_k}{10^k} = 0.x_1 + 0.0x_2 + 0.00x_3 + \ldots$$

Thus, we can similarly express

$$x = \sum_{k=1}^{\infty} \frac{x_k}{3^k},$$

in ternary representation. However, just as

$$0.99999\ldots \text{ and } 1.00000\ldots$$

are indistinguishable, in ternary representation,

$$0.22222\ldots \text{ and } 1.00000\ldots$$

are indistinguishable. Fortunately, we can find out who exactly are the culprits that cannot be uniquely represented, which shall be left as an exercise.

I shall paraphrase the professor here because I like how the analogy brings good intuition, for me at least.

> *Suppose there's this person that had only 3 fingers and is not aware of the existence of the base-10 system, and in turn invented the ternary system. Then, instead of having 10 regular intervals on $[0, 1]$, it had 3 regular intervals.*

Exercise 5.1.1

Show that the ternary expansion of $x \in [0, 1)$ is unique except when $\exists N \geq 1$ such that

$$x = \frac{r}{3^N},$$

for some $0 < r < 3^N$, where $3 \nmid r$.

In the cases where we have the above x, we have that [3]

$$x = 0.x_1x_2x_3\ldots x_N,$$

where $x_N \in \{1, 2\}$.

- If $x_N = 2$, we shall keep this expression; otherwise
- if $x_N = 1$, then we write

$$\begin{aligned} x &= 0.x_1x_2x_3\ldots x_{N-2}x_{N-1}1000\ldots \\ &= 0.x_1x_2x_3\ldots x_{N_2}x_{N-1}0222\ldots, \end{aligned}$$

and we shall use the second expression.

[3] Note that the representation terminates somewhere, since it is a fraction, i.e. a rational number.

Also, we shall also use the convention that

$$1 = 0.22222\ldots.$$

With this, we have obtained a **unique** ternary expansion for each $x \in [0,1]$.

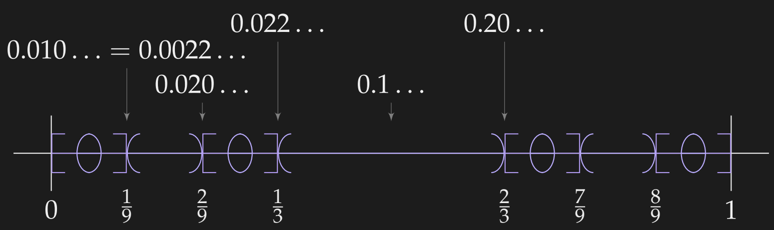

Figure 5.3: Some values on $[0,1]$ in ternary representation

Now, observe that

$$\begin{aligned} C_1 &= [0,1] \setminus \left(\frac{1}{3}, \frac{2}{3} \right) \\ &= \{ x \in [0,1] : x = 0.x_1x_2x_3\ldots, x_1 \neq 1 \}, \end{aligned}$$

i.e. whichever $x \in [0,1]$ with $x_1 = 1$ sits in $\left(\frac{1}{3}, \frac{2}{3}\right)$. Similarly,

$$C_2 = \{ x \in [0,1] : x = 0.x_1x_2x_3\ldots, x_1 \neq 1, x_2 \neq 1 \}.$$

In general, we have that

$$C_N = \{ x \in [0,1] : x = 0.x_1x_2x_3\ldots, x_i \neq 1, 1 \leq i \leq N \}.$$

Therefore,

$$\begin{aligned} C &= \bigcap_{n=1}^{\infty} C_n \\ &= \{ x \in [0,1] : x = 0.x_1x_2x_3\ldots, x_n \neq 1, n \geq 1 \} \\ &= \{ x \in [0,1] : x = 0.x_1x_2x_3\ldots, x_n \in \{0,2\}, n \geq 1 \} \end{aligned}$$

Now, consider the bijection

$$\varphi : C \to [0,1]$$

given by

$$x = 0.x_1x_2x_3\ldots \mapsto y = 0.y_1y_2y_3\ldots,$$

where $x_n \in \{0, 2\}$, for $n \geq 1$, and x is the ternary expansion, while $y_n = \frac{x_n}{2}$ for each $n \geq 1$, and so y is a binary expansion. Then φ is a bijection between C and $[0, 1]$, and therefore

$$|C| = |[0,1]| = |\mathbb{R}| = c = 2^{\aleph_0}.$$

Note 5.1.1

The lesson here is that the Lebesgue measure is not a measure on the cardinality of the set. Rather, it measures the ***distribution of points*** *in the set.*

5.2 Lebesgue Measurable Functions

Note 5.2.1

We used

$$\mathfrak{M}(\mathbb{R}) = \{E \subseteq \mathbb{R} \mid E \text{ is measurable } \}$$

to denote the set of measurable subsets of $\mathbb{R}$*.*

In general, for $H \subseteq \mathbb{R}$*, set shall denote by* $\mathfrak{M}(H)$ *the collection of all Lebesgue measurable subsets of* H*, i.e.*

$$\mathfrak{M}(H) = \{E \subseteq H \mid E \in \mathfrak{M}(\mathbb{R})\}.$$

In particular, for $E \in \mathfrak{M}(\mathbb{R})$*, we also have*

$$\mathfrak{M}(E) = \{F \subseteq E \mid F \in \mathfrak{M}(\mathbb{R})\}.$$

Exercise 5.2.1

Prove that the above $\mathfrak{M}(E)$ *is a* σ*-algebra of sets.*

Definition 20 (Lebesgue Measurable Function)

Let $E \in \mathfrak{M}(E)$ and (X, d) a metric space. We say that a function

$$f : E \to X$$

*is **Lebesgue measurable** (or simply **measurable**) if*

$$f^{-1}(G) ::= \{x \in E : f(x) \in G\} \in \mathfrak{M}(E)$$

for every open set $G \subseteq X$.

We write

$$\mathcal{L}(E, X) = \{f : E \to X \mid f \text{ measurable }\}$$

for the set of measurable functions from E to X.

Exercise 5.2.2

Show that we can equivalently define that a function f is Lebesgue measurable if

$$f^{-1}(F) \in \mathfrak{M}(E)$$

for all closed subsets $F \subseteq X$.

Note 5.2.2

Note that we required that the domain of the function is a measurable set in Definition 20. Part of the reason is because we want constant functions to be measurable, and this happens iff the domain of the function is measurable [4].

[4] **Why?**

Proposition 19 (Continuous Functions on a Measurable Set is Measurable)

Let $E \in \mathfrak{M}(\mathbb{R})$ and (X, d) a metric space. If $f : E \to X$ is continuous, then $f \in \mathcal{L}(E, X)$.

Proof

Since f is continuous in a metric space, it implies that for all open $G \subseteq X$, $f^{-1}(G)$ is open in E [5]. This means that $f^{-1}(G) = U_G \cap E$ for some open $U_G \subseteq \mathbb{R}$. Since U_G is open, by Proposition 14, $U_G \in \mathfrak{M}(\mathbb{R})$. Since $E \in \mathfrak{M}(\mathbb{R})$, we have that

$$f^{-1}(G) = U_G \cap E \in \mathfrak{M}(E),$$

and so

$$f \in \mathcal{L}(E, X).$$

□

[5] We say that $f^{-1}(G)$ is **relatively open** in E.

Example 5.2.1

Let $E \in \mathfrak{M}(\mathbb{R})$ and $H \subseteq E$. Consider the characteristic function of H, which is

$$\chi_H : E \to \mathbb{R} \text{ given by } x \mapsto \begin{cases} 1 & x \in H \\ 0 & x \notin H \end{cases}.$$

Let $G \subseteq \mathbb{R}$ be open. Then

$$\chi_H^{-1}(G) = \begin{cases} \emptyset & G \cap \{0,1\} = \emptyset \\ E & G \supseteq \{0,1\} \\ E \setminus H & G \cap \{0,1\} = \{0\} \\ H & G \cap \{0,1\} = \{1\} \end{cases},$$

in which case we observe that all the possible outcomes are measurable subsets of $\mathbb{R}$. Thus χ_H is measurable iff $H \in \mathfrak{M}(\mathbb{R})$.

Proposition 20 (Composition of a Continuous Function and a Measurable Function is Measurable)

Let $E \in \mathfrak{M}(\mathbb{R})$ and (X, d_X), (Y, d_Y) be metric spaces. Suppose that

$$f : E \to X \text{ is measurable and } g : X \to Y \text{ is continuous.}$$

Then

$$g \circ f : E \to Y \text{ is measurable.}$$

The idea is simple: $(gf)^{-1}(G) = f^{-1}g^{-1}(G)$ and continuity of G means that $g^{-1}(G)$ is open in X.

Proof

Let $G \subseteq Y$ be open. Then since g is continuous, we have that

$$g^{-1}(G) \subseteq X \text{ is open.}$$

Then since f measurable, we have that

$$(g \circ f)^{-1}(G) = f^{-1}(g^{-1}(G)) \in \mathfrak{M}(E).$$

Thus $g \circ f \in \mathcal{L}(E, Y)$. □

Example 5.2.2

Let $E \in \mathfrak{M}(E)$ and $f \in \mathcal{L}(E, \mathbb{K})$. Let $g : \mathbb{K} \to \mathbb{R}$ be given by $z \mapsto |z|$. Then g is continuous. By Proposition 20, we have that

$$g \circ f = |f| \text{ is measurable.}$$

Example 5.2.3

Note that the converse to the above is not true, i.e. that if we have that $|f|$ is measurable, it is not necessary that f is measurable.

Consider $E = \mathbb{R} = \mathbb{K}$. If we take $H \subseteq \mathbb{R}$ that is not measurable, which we know exists, and then consider the function

$$f : E \to \mathbb{R} \text{ given by } f(x) = \begin{cases} 1 & x \in H \\ -1 & x \notin H \end{cases},$$

which is constructed by summing up two characteristic functions over H and then minus 1. Then $|f| = 1$, but

$$f^{-1}(\{1\}) = H \notin \mathfrak{M}(\mathbb{R}).$$

Proposition 21 (Component-wise Measurability)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $f, g : E \to \mathbb{K}$. Then TFAE:

1. $f, g \in \mathcal{L}(E, \mathbb{K})$;
2. $h : E \to \mathbb{K}^2$ *given by* $x \mapsto (f(x), g(x))$ *is measurable.*

Proof

$(2) \implies (1)$ [6] Let

$$\pi_1 : \mathbb{K}^2 \to \mathbb{K} \text{ given by } (w, z) \mapsto w$$
$$\pi_2 : \mathbb{K}^2 \to \mathbb{K} \text{ given by } (w, z) \mapsto z$$

so that π_1, π_2 are continuous. Then by Proposition 20, we have that

$$\pi_1 \circ h = f \text{ and } \pi_2 \circ h = g$$

are both measurable.

$(1) \implies (2)$ Let $G \subseteq \mathbb{K}^2$ be open. We can write G as a countable union of open sets [7], i.e.

$$G = \bigcup_{n=1}^{\infty} A_n \times B_n,$$

where $A_n, B_n \subseteq \mathbb{K}$ are open. Then

$$\begin{aligned} h^{-1}(G) &= h^{-1}\left(\bigcup_{n=1}^{\infty} A_n \times B_n \right) \\ &= \bigcup_{n=1}^{\infty} \underbrace{f^{-1}(A_n)}_{\in \mathfrak{M}(\mathbb{K})} \cap \underbrace{g^{-1}(B_n)}_{\in \mathfrak{M}(\mathbb{K})} \in \mathfrak{M}(\mathbb{K}) \end{aligned}$$

Thus $h \in \mathcal{L}(E, \mathbb{K}^2)$. □

[6] Awareness about projective maps is a plus here.

[7] If you are unsure about this, think $\varepsilon - \delta$.

Proposition 22 ($\mathcal{L}(E, \mathbb{K})$ is a Unital Algebra)

*Let $E \in \mathfrak{M}(\mathbb{R})$. Then $\mathcal{L}(E, \mathbb{K})$ is a **unital algebra**, i.e. if $f, g \in \mathcal{L}(E, \mathbb{K})$, then*

1. $f + g \in \mathcal{L}(E, \mathbb{K})$;

2. $fg \in \mathcal{L}(E, \mathbb{K})$ [8];
3. $g(x) \neq 0, \forall x \in E \implies \frac{f}{g} \in \mathcal{L}(E, \mathbb{K})$; *and*
4. *if* $h : E \to \mathbb{K}$ *is constant, then* $h \in \mathcal{L}(E, \mathbb{K})$.

In particular, $\mathcal{L}(E, \mathbb{K})$ *is an* ***algebra****.*

[8] Here, it's multiplication of two functions, not compositions

Proof

We shall make use of this clever trick [9]. Let $\mu : E \to \mathbb{K}^2$ given by $x \mapsto (f(x), g(x))$. Note that since $f, g \in \mathcal{L}(E, \mathbb{K})$, by Proposition 21, $\mu \in \mathcal{L}(E, \mathbb{K}^2)$.

[9] "Clever trick" = "Trick you should learn".

1. Consider the function

$$\sigma : \mathbb{K}^2 \to \mathbb{K} \text{ given by } (w, z) \mapsto w + z.$$

   It is clear that σ is continuous. Then

$$\sigma \circ \mu : x \mapsto f(x) + g(x)$$

   is measurable by Proposition 20.

2. Consider the function

$$\sigma : \mathbb{K}^2 \to \mathbb{K} \text{ given by } (w, z) \mapsto wz.$$

   Again, we see that σ is continuous. Then

$$\sigma \circ \mu : x \mapsto f(x)g(x)$$

   is measurable by Proposition 20.

3. Consider the function

$$\sigma : \mathbb{K} \times (\mathbb{K} \setminus \{0\}) \to \mathbb{K} \text{ given by } (w, z) \mapsto \frac{w}{z}.$$

   Again, σ is continuous. Thus

$$\sigma \circ \mu : x \mapsto \frac{f(x)}{g(x)}$$

is measurable by Proposition 20.

4. Suppose $h : E \to \mathbb{K}$ is a constant, and we have $h(x) = \alpha_0$ for all $x \in E$. Then for any $G \subseteq \mathbb{K}$ that is open, we have that

$$h^{-1}(G) = \begin{cases} \emptyset & a_0 \notin G \\ E & a_0 \in G \end{cases},$$

both of which are measurable sets. Thus h is indeed measurable.

□

Warning (Composition of Measurable Functions Need Not be Measurable)

It is important to note that compositions of measurable functions do not have to be measurable. Here is a counterexample [10].

Let $f : [0,1] \to [0,1]$ be the ***Cantor-Lebesgue Function*** [11]. *Note that f is a monotonic and continuous function, and the image $f(C)$ of the Cantor set C is all of $[0,1]$. Let $g(x) = x + f(x)$. It is clear that $g : [0,1] \to [0,2]$ is a strictly monotonic and continuous map. In particular, $h = g^{-1}$ is also continuous.*

[10] Source: Mirjam 2013

[11] Seen in A2Q5.

Remark 5.2.1

Note that $(\mathbb{C}, d)$, where $d(w,z) = |w - z|$, is a metric space. Moreover, the map

$$\gamma : \mathbb{C} \to \mathbb{R}^2 \text{ given by } x + iy \mapsto (x, y),$$

where $x, y \in \mathbb{R}$ is a homeomorphism, which, in particular, is continuous. Then given a $E \in \mathfrak{M}(\mathbb{R})$ with a measurable $f \in E \to \mathbb{C}$, then

$$\gamma \circ f : E \to \mathbb{R}^2 \in \mathcal{L}(E, \mathbb{R}^2).$$

Also, notice that

$$\gamma \circ f = (\Re f, \Im f).$$

By Proposition 21, $\Re f, \Im f \in \mathcal{L}(E, \mathbb{R})$. *This also means that*

$$h : x \mapsto (\Re f(x), \Im f(x)) \in \mathcal{L}(E, \mathbb{R}^2).$$

Conversely, if $\Re f, \Im f \in \mathcal{L}(E, \mathbb{R})$, *then*

$$f = \gamma^{-1} \circ h \in \mathcal{L}(E, \mathbb{C})$$

by Proposition 21.

This means that a ***complex-valued function is measurable iff its real and imaginary parts are both measurable****. Consequently, to study about complex-valued functions, it is sufficient for us to study about real-valued functions.*

Proposition 23 (Measurable Function Broken Down into an Absolute Part and a Scaling Part)

Let $E \in \mathfrak{M}(\mathbb{R})$ *and suppose that* $f : E \to \mathbb{C}$ *is measurable. Then there exists a measurable function* $\Theta : E \to \mathbb{T}$, *where*

$$\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\},$$

such that

$$f = \Theta \cdot |f|.$$

Proof

Since $\{0\} \subseteq \mathbb{C}$ is closed and f is measurable, we have that

$$K := f^{-1}(\{0\}) \in \mathfrak{M}(E).$$

Since χ_K is a measurable function, we have that $f + \chi_K$ is also measurable (cf. Proposition 22).

Claim: $f + \chi_K \neq 0$ over E.

- If $x \in E$ such that $f(x) = 0$, then $x \in K$, and so $\chi_K(x) = 1$.

- If $x \in E$ such that $\chi_K(x) = 0$, then $x \notin K$, which means $f(x) \neq 0$.

  Therefore, consider the function

$$\Theta = \frac{f + \chi_K}{|f + \chi_K|} : E \to \mathbb{T}.$$

By Proposition 22, Θ is measurable, and clearly

$$f = \Theta \cdot |f| \, . \qquad \square$$

Remark 5.2.2

As of now, given a set $E \in \mathfrak{M}(\mathbb{R})$, to verify that a function $f \in \mathcal{L}(E, \mathbb{R})$, we need to check that

$$\forall G \subseteq \mathbb{R} \text{ open }, \, f^{-1}(G) \in \mathfrak{M}(E).$$

Since there is an obscene amount of open (respectively closed) subsets of $\mathbb{R}$, we want to be able to reduce our workload. This shall be the first thing we do in the next lecture.

6 Lecture 6 May 23rd 2019

6.1 Lebesgue Measurable Functions (Continued)

Proposition 24 (Function Measurability Check)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $f : E \to \mathbb{R}$ be a function. TFAE:

1. f *is measurable, i.e.* $\forall G \subseteq \mathbb{R}$ *that is open,* $f^{-1}(G) \in \mathfrak{M}(E)$.
2. $\forall a \in \mathbb{R}$, $f^{-1}((a, \infty)) \in \mathfrak{M}(E)$.
3. $\forall b \in \mathbb{R}$, $f^{-1}((-\infty, b]) \in \mathfrak{M}(E)$.
4. $\forall b \in \mathbb{R}$, $f^{-1}((-\infty, b)) \in \mathfrak{M}(E)$.
5. $\forall a \in \mathbb{R}$, $f^{-1}([a, \infty)) \in \mathfrak{M}(E)$.

Proof

(1) $\implies$ (2) This is trivially true since $\forall a \in \mathbb{R}$, (a, ∞) is open in $\mathbb{R}$, and so since f is measurable, we must have that $f^{-1}((a, \infty)) \in \mathfrak{M}(E)$.

(2) $\implies$ (3) Notice that $\forall b \in \mathbb{R}$,

$$f^{-1}((-\infty, b]) = f^{-1}(\mathbb{R} \setminus (b, \infty)) = E \setminus f^{-1}((b, \infty))$$

and $f^{-1}((b, \infty)) \in \mathfrak{M}(E)$ by assumption. Since $\mathfrak{M}(E)$ is a σ-algebra, $f^{-1}((-\infty, b]) \in \mathfrak{M}(E)$.

(3) $\implies$ (4) Notice that $\forall b \in \mathbb{R}$,

$$f^{-1}((-\infty, b)) = \bigcup_{n=1}^{\infty} f^{-1}\left(\left(-\infty, b - \frac{1}{n}\right]\right),$$

and by assumption, for each $n \geq 1$, $f^{-1}\left(\left(-\infty, b - \frac{1}{n}\right]\right) \in \mathfrak{M}(E)$. It follows that $f^{-1}((-\infty, b)) \in \mathfrak{M}(E)$.

(4) $\implies$ (5) Observe that $\forall a \in \mathbb{R}$, we have

$$f^{-1}([a, \infty)) = f^{-1}(\mathbb{R} \setminus (-\infty, a)) \in \mathfrak{M}(E)$$

by assumption.

(5) $\implies$ (1) [1] Notice that $\forall a \in \mathbb{R}$,

$$f^{-1}((a, \infty)) = \bigcup_{n=1}^{\infty} f^{-1}\left(\left[a + \frac{1}{n}, \infty\right)\right) \in \mathfrak{M}(E)$$

by assumption. Furthermore, we have that $\forall b \in \mathbb{R}$,

$$f^{-1}((-\infty, b)) = E \setminus f^{-1}([b, \infty)) \in \mathfrak{M}(E),$$

also by assumption. Thus

$$f^{-1}((a, b)) = f^{-1}((a, \infty)) \cap f^{-1}((-\infty, b)) \in \mathfrak{M}(E),$$

for any $a, b \in \mathbb{R}$.

Since for any open $G \subseteq \mathbb{R}$ can be written as a countable union of open intervals, i.e.

$$G = \bigcup_{n=1}^{\infty} I_n,$$

where each I_n is an open interval, we have that

$$f^{-1}(G) = \bigcup_{n=1}^{\infty} f^{-1}(I_n) \in \mathfrak{M}(E).$$

Thus f is measurable. □

[1] This uses the same idea as in Proposition 14.

The proof of the following result is left to A2.

Corollary 25 (Measurability Check on the Borel Set)

If $E \in \mathfrak{M}(\mathbb{R})$ and $f : E \to \mathbb{R}$ is a function, then TFAE:

1. *f is measurable.*
2. $\forall B \in \mathfrak{Bor}(\mathbb{R})$, $f^{-1}(B) \in \mathfrak{M}(E)$.

Remark 6.1.1

Let $E \in \mathfrak{M}(\mathbb{R})$ and $f : E \to \mathbb{R}$. Define

$$f^+(x) = \max\{f(x), 0\},\ x \in E$$
$$f^-(x) = \max\{-f(x), 0\},\ x \in E$$

Then $f^+, f^- \geq 0$, and

$$f = f^+ - f^- \text{ and } |f| = f^+ + f^-.$$

Moreover,

$$f^+ = \frac{|f| + f}{2} \text{ and } f^- = \frac{|f| - f}{2},$$

and so both f^+ and f^- are measurable.

By Remark 5.2.1, every complex-valued measurable function is a linear combination of 4 non-negative, real-valued measurable functions.

We shall now examine a number of results dealing with pointwise limits of sequences of measurable, real-valued functions. We shall include the case where the limit of a given point is allowed to be an **extended real number**; i.e. the sequence diverges either to ∞ or $-\infty$.

Definition 21 (Extended Real Numbers)

*We define the **extended real numbers** to be the set*

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}.$$

We also write $\overline{\mathbb{R}} = [-\infty, \infty]$.

By convention, we shall define

- $\infty + \infty = \infty$, $-\infty - \infty = -\infty$;
- $\forall \alpha \in \mathbb{R} \cup \{\infty\}$, $\alpha + \infty = \infty = \infty + \alpha$;
- $\forall \alpha \in \mathbb{R}$, $\alpha + (-\infty) = -\infty = -\infty + \alpha$;
- $\forall 0 < \alpha \in \overline{\mathbb{R}}$, $a \cdot \infty = \infty \cdot \alpha = (-\infty) \cdot (-\alpha) = (-\alpha) \cdot (-\infty) = \infty$;
- $\forall \alpha < 0 \in \overline{\mathbb{R}}$, $a \cdot \infty = \infty \cdot \alpha = (-\infty) \cdot (-\alpha) = (-\alpha) \cdot (-\infty) = -\infty$; *and*
- $0 = 0 \cdot \infty = \infty \cdot 0 = 0 \cdot (-\infty) = (-\infty) \cdot 0$.

Warning

Notice that we do not define $\infty - \infty$ and $-\infty + \infty$.

Note 6.1.1

While the space of extended real numbers is useful for treating measure-theoretic and analytic properties of sequences of functions, it has poor algebraic properties. In particular, it is no longer a vector space, since ∞ and $-\infty$ do not have their additive inverses.

Definition 22 (Extended Real-Valued Function)

*Given $H \subseteq \mathbb{R}$, the function $f : H \to \overline{\mathbb{R}}$ is called an **extended real-valued function**.*

Definition 23 (Measurable Extended Real-Valued Function)

*If $E \in \mathfrak{M}(\mathbb{R})$ and $f : E \to \overline{\mathbb{R}}$ is an extended real-valued function, we say that f is **Lebesgue measurable** (or simply **measurable**) if*

1. $\forall G \subseteq \mathbb{R}$ *open*, $f^{-1}(G) \in \mathfrak{M}(E)$ *; annd*
2. $f^{-1}(\{-\infty\}), f^{-1}(\{\infty\}) \in \mathfrak{M}(E)$.

We denote the set of Lebesgue measurable extended real-valued functions on E by

$$\mathcal{L}(E, \overline{\mathbb{R}}) = \{ f : E \to \overline{\mathbb{R}} : f \text{ is measurable } \}.$$

Since we shall often refer to only the non-negative elements of $\mathcal{L}(E, \overline{\mathbb{R}})$, we also define the notation

$$\mathcal{L}(E, [0, \infty]) = \{ f \in \mathcal{L}(E, \overline{\mathbb{R}}) : \forall x \in E,\, 0 \leq f(x) \}.$$

Note 6.1.2

Note that we can also replace the first condition of Lebesgue measurability of extended real-valued functions by

$$\forall F \subseteq \mathbb{R} \text{ closed },\, f^{-1}(F) \in \mathfrak{M}(E).$$

Just as in the case with regular real-valued measurable functions, we have the following shortcuts in testing whether an extended real-valued function is measurable.

Notation

We write

- $(a, \infty] = (a, \infty) \cup \{\infty\}$; *and*
- $[-\infty, b) = (-\infty, b) \cup \{-\infty\}$,

for all $a, b \in \mathbb{R}$.

Proposition 26 (Measurability Check for Extended Real-Valued Functions)

Let $E \in \mathfrak{M}(\mathbb{R})$ and suppose $f : E \to \overline{\mathbb{R}}$ is a function. Then TFAE:

1. *f is Lebesgue measurable.*

2. $\forall a \in \mathbb{R}$, $f^{-1}((a, \infty]) \in \mathfrak{M}(E)$.
3. $\forall b \in \mathbb{R}$, $f^{-1}([-\infty, b)) \in \mathfrak{M}(E)$.

Exercise 6.1.1

Prove Proposition 26.

Proposition 27 (Measurability of Limits and Extremas)

Let $E \in \mathfrak{M}(\mathbb{R})$ and suppose that $(f_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{L}(E, \overline{\mathbb{R}})$. Then the following extended real-valued functions are also measurable:

1. $g_1 := \sup_{n \geq 1} f_n$;
2. $g_2 := \inf_{n \geq 1} f_n$;
3. $g_3 := \limsup_{n \geq 1} f_n$; *and*
4. $g_4 := \liminf_{n \geq 1} f_n$.

Proof

1. Let $a \in \mathbb{R}$. Then

$$g_1^{-1}((a, \infty]) = \bigcup_{n \geq 1} \underbrace{f_n^{-1}((a, \infty])}_{\in \mathfrak{M}(E)} \in \mathfrak{M}(E).$$

It follows from Proposition 26 that $g_1 \in \mathcal{L}(E, \overline{\mathbb{R}})$.

2. [2] For any $b \in \mathbb{R}$, we have

$$g_2^{-1}([-\infty, b)) = \bigcap_{n \geq 1} f_n^{-1}([-\infty, b)) \in \mathfrak{M}(E).$$

Thus by Proposition 26, $g_2 \in \mathcal{L}(E, \overline{\mathbb{R}})$.

3. Let $h_n = \sup_{k \geq n} f_n$ for each $n \geq 1$. Then by part (1), $h_n \in \mathcal{L}(E, \overline{\mathbb{R}})$ for each $n \geq 1$. Also, notice that $h_1 \geq h_2 \geq h_3 \geq \ldots$, i.e. $\{h_n\}_{n=1}^{\infty}$

[2] **Both notes and lecture notes used union, but should it not be intersection?**

is an increasing sequence of functions. Then by part (2),

$$g_3 = \lim_{n\to\infty} h_n = \inf_{n\geq 1} h_n \in \mathcal{L}(E, \overline{\mathbb{R}}).$$

4. Let $h_n = \inf_{k\geq n} f_n$ for each $n \geq 1$. Then by part (2), each $h_n \in \mathcal{L}(E, \overline{\mathbb{R}})$. Also, $\{h_n\}_{n=1}^{\infty}$ is a decreasing sequence of functions. Then by part (1), we have that

$$g_4 = \lim_{n\to\infty} h_n = \sup_{n\geq 1} h_n \in \mathcal{L}(E, \overline{\mathbb{R}}).$$ □

Corollary 28 (Extended Limit of Real-Valued Functions)

Let $E \in \mathfrak{M}(\mathbb{R})$ and suppose that $(f_n)_{n=1}^{\infty}$ is a sequence of real-valued functions such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists as an extended real-valued number for all $x \in E$. Then

$$f \in \mathcal{L}(E, \overline{\mathbb{R}}).$$

Proof

By A2, when the said limit exists, we have that

$$f = \limsup_{n\geq 1} f_n = \liminf_{n\geq 1} f_n,$$

and so $f \in \mathcal{L}(E, \overline{\mathbb{R}})$ by Proposition 27. □

Definition 24 (Simple Functions)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $\varphi : E \to \overline{\mathbb{R}}$. We say that φ is ***simple*** *if* range φ *is finite. Furthermore, we denote the set of all simple, real-valued,* ***measurable*** *functions on E as*

$$\mathrm{SIMP}(E, \mathbb{R}).$$

Definition 25 (Standard Form)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $\varphi : E \to \overline{\mathbb{R}}$. Suppose that

$$\text{range}\,\varphi = \{\alpha_1 < \alpha_2 < \ldots < \alpha_N\},$$

and set

$$E_n := \varphi^{-1}(\{\alpha_n\}), \text{ for } 1 \leq n \leq N.$$

We say that

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{E_n}$$

*is the **standard form** of φ.*

Warning (Step Functions are Simple, but the Converse is False)

*Recall that a **step function** is a function that can be written as a finite linear combination of indicator functions of intervals. This means that step functions are simple functions. However, simple functions are not necessarily step functions. For example, χ_C, where C is the Cantor set, is a simple function since C is measurable, but it is clearly not a step function, as it would require infinitely many indicator functions of infinitely small intervals.*

Proposition 29 (Measurability of Simple Functions with Measurable Support)

Let $E \in \mathfrak{M}(\mathbb{R})$. Suppose $\varphi : E \to \overline{\mathbb{R}}$ is simple with

$$\text{range}\,\varphi = \{\alpha_1 < \alpha_2 < \ldots < \alpha_N\}.$$

TFAE:

1. *φ is measurable.*
2. *If $\varphi = \sum_{n=1}^{N} \alpha_n \chi_{E_n}$ is the standard form of φ, then $E_n \in \mathfrak{M}(E)$, for all $n \in \{1, \ldots, N\}$.*

Proof

($\implies$) Since φ is measurable, notice that for each $n \in \{1, \ldots, N\}$,

- if $\alpha_n \in \mathbb{R}$, then $\{\alpha_n\}$ is closed, and so

$$E = \varphi^{-1}(\{\alpha_n\}) \in \mathfrak{M}(E); \text{ and}$$

- if $\alpha_1 = -\infty$, and similarly if $\alpha_N = \infty$, then by Definition 23, $\varphi^{-1}(\{\alpha_1\}), \varphi^{-1}(\{\alpha_N\}) \in \mathfrak{M}(E)$.

($\impliedby$) By Example 5.2.1, $\forall n \geq 1$, $E_n \in \mathfrak{M}(E) \implies \forall n \geq 0 \chi_{E_n} \in \mathfrak{M}(E)$. Notice that $\forall a \in \mathbb{R}$,

$$\varphi^{-1}((a, \infty]) = \bigcup \{E_n : a < \alpha_n\},$$

and so $\varphi^{-1}((a, \infty])$ is a finite (or empty) union of measurable sets, and is hence measurable. □

The standard form is not a unique way of expressing a simple function as a finite linear combination of characteristic functions.

Example 6.1.1

Consider the function $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi = \chi_{\mathbb{Q}} + 9\chi_{[2,6]}.$$

Then range $\varphi = \{0, 1, 9, 10\}$; we see that

$$x \mapsto \begin{cases} 0 & x \in \mathbb{Q}^C \cap [2,6]^C \\ 1 & x \in \mathbb{Q} \cap [2,6]^C \\ 9 & x \in \mathbb{Q}^C \cap [2,6] \\ 10 & x \in \mathbb{Q} \cap [2,6] \end{cases}.$$

Then we may write φ as

$$\varphi = 0\chi_{\mathbb{Q}^C \cap [2,6]^C} + 1\chi_{\mathbb{Q} \cap [2,6]^C} + 9\chi_{\mathbb{Q}^C \cap [2,6]} + 10\chi_{\mathbb{Q} \cap [2,6]}.$$

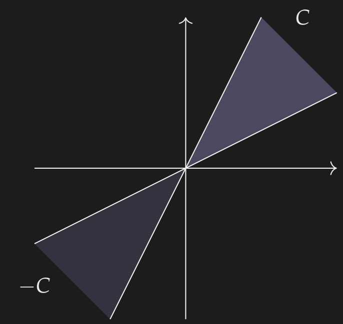

Figure 6.1: Typical figure of a cone

Definition 26 (Real Cone)

Let $\mathcal{V}$ be a vector space over $\mathbb{K}$. A subset $C \subseteq \mathcal{V}$ is said to be a ***(real) cone*** *is*

1. $C \cap -C = \{0\}$, *where* $-C = \{-w : w \in C\}$; *and*
2. $y, z \in C$ *and* $0 \leq \kappa \in \mathbb{R}$ *imply that*

$$\kappa y + z \in C.$$

Example 6.1.2

1. Let $\mathcal{V} = \mathbb{R}^3$ and

$$C = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x, y, z\}.$$

   Then C is a (real) cone.

2. Let $\mathcal{V} = \mathbb{C}$ and

$$C = \left\{ w \in \mathbb{C} : w = re^{i\theta}, \frac{\pi}{6} \leq \theta \leq \frac{2\pi}{6}, 0 \leq r < \infty \right\}.$$

   The C is a cone in $\mathbb{C}$. Note that in both the above examples, C is not closed.

3. Let $\mathcal{V} = \mathcal{C}([0, 1], \mathbb{C})$, and

$$C = \{f \in \mathcal{V} : 0 \leq f(x), \forall x \in [0, 1]\},$$

   where we note that the condition means that we only look at those functions that return real positive values. Then C is a (real) cone in $\mathcal{V}$.

Exercise 6.1.2

Show that SIMP$(E, \mathbb{R})$ *is an algebra, and hence a vector space over* $\mathbb{R}$.

Remark 6.1.2

1. *Note that*

$$\mathrm{SIMP}(E, \overline{\mathbb{R}}) = \{ f : E \to \overline{\mathbb{R}} : f \text{ is simple and measurable } \}.$$

   is not a vector space. In fact, it is neither a field nor a ring.

2. *We shall adopt the following notation:*

$$\mathrm{SIMP}(E, [0, \infty)) := \{ \varphi \in \mathrm{SIMP}(E, \mathbb{R}) : 0 \leq \varphi(x) \text{ for all } x \in E \}.$$

   Observe that this is a real cone in $\mathrm{SIMP}(E, \mathbb{R})$.

   In A3, we will show the following proposition.

Proposition 30 (Increasing Sequence of Simple Functions that Converges to a Measurable Function)

Let $E \in \mathfrak{M}(E)$ *and* $f \in \mathcal{L}(E, [0, \infty])$*. Then there exists an increasing sequence*

$$\varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \ldots \leq f$$

of simple, real-valued functions φ_n *such that*

$$f(x) = \lim_{n \to \infty} \varphi_n(x)$$

for all $x \in E$.

7 Lecture 7 May 28th 2019

7.1 Lebesgue Integration

We shall first begin by defining integration over simple, non-negative, extended real-valued functions. We shall then use this definition to define the integral of $f \in \mathcal{L}(E, [0, \infty])$, and derive several consequences of our definition. Furthermore, we shall also build the Lebesgue integral such that it is linear, which will require us to impose certain conditions to the range of functions which will retain this desirable property.

Definition 27 (Integration of Simple Functions)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $\varphi \in \mathrm{SIMP}(E, [0, \infty])$, such that its standard form is denoted as

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{E_n}.$$

We define

$$\int_E \varphi := \sum_{n=1}^{N} \alpha_n m E_n \in [0, \infty].$$

If $F \subseteq E$ is measurable, we define

$$\int_F \varphi = \int_E \varphi \cdot \chi_F = \sum_{n=1}^{N} \alpha_n m(F \cap E_n).$$

Note 7.1.1

Note that since φ is measurable, so is each E_n for $1 \leq n \leq N$.

Example 7.1.1

1. Let $\varphi = 0\chi_{[4,\infty)} + 17\chi_{\mathbb{Q}\cap[0,4)} + 29\chi_{[2,4)\setminus\mathbb{Q}}$. Then

$$\begin{aligned}\int_{[0,\infty)} \varphi &= 0m[4,\infty) + 17m(\mathbb{Q}\cap[0,4)) + 29m([2,4)\setminus\mathbb{Q}) \\ &= 0 + 17\cdot 0 + 29(2) = 58.\end{aligned}$$

2. Let $C \subseteq [0,1]$ be the Cantor set from Example 5.1.1 and $\varphi = 1\chi_C + 2\chi_{[5,9]}$. Then

$$\begin{aligned}\int_{[0,6]} \varphi &= 1m(C\cap[0,6]) + 2m([5,9]\cap[0,6]) \\ &= 1\cdot 0 + 2m([5,6]) \\ &= 2.\end{aligned}$$

Since our definition is fairly limited since it requires that our simple function be in standard form, let us try to relax that condition.

Definition 28 (Disjoint Representation)

Let $E \in \mathfrak{M}(E)$ and $\varphi \in \mathrm{SIMP}(E,[0,\infty])$. Suppose

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{H_n},$$

*where $H_n \subseteq E$ is **measurable** and $\alpha_n \in \overline{\mathbb{R}}$ for each $1 \leq n \leq N$.* [1] *We shall say that the above decomposition of φ is a **disjoint representation** of φ if*

$$H_i \cap H_j = \emptyset, \text{ for } 1 \leq i \neq j \leq N.$$

[1] Note that we did not require that the α_n's be distinct, nor do we require that they be written in any particular order, nor do we require that $E = \bigcup_{n=1}^{N} H_n$, unlike in the definition of simple functions.

Lemma 31 (Common Disjoint Representation of Simple Functions over a Common Domain)

Let $E \in \mathfrak{M}(\mathbb{R})$ and suppose that $\varphi, \psi \in \mathcal{L}(E,\mathbb{R})$. Then there exists

1. $N \in \mathbb{N}$ *;*

2. *$H_1, H_2, \ldots, H_n \in \mathfrak{M}(E)$ with $H_i \cap H_j = \emptyset$ for all $i \neq j$; and*

3. *$\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N$ such that*

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{H_n} \quad \text{and} \quad \psi = \sum_{n=1}^{N} \beta_N \chi_{H_n}$$

are disjoint representations of φ and ψ.

Proof

Since φ and ψ are simple, from Definition 25, if we write

$$\varphi = \sum_{m=1}^{M_1} a_m \chi_{E_m} \quad \text{and} \quad \psi = \sum_{m=1}^{M_2} b_m \chi_{F_m}$$

in their standard forms, we have that the E_m's and F_m's are respectively pairwise disjoint [2]. Then

[2] It is important to note here that the E_m's and F_m's are pairwise disjoint on E, which is why the next step is a sensible and correct one.

$$\{E_i \cap F_j : 1 \leq i \leq M_1,\, 1 \leq j \leq M_2\}$$

is also a pairwise disjoint family of measurable sets, which we shall relabel them as

$$\{H_n\}_{n=1}^{N}, \text{ where } N = M_1 M_2.$$

Then

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{H_n},$$

where $\alpha_n = a_i$ if $H_n = E_i \cap F_j$ for some $1 \leq j \leq M_2$, and

$$\psi = \sum_{n=1}^{N} \beta_N \chi_{H_n},$$

where $\beta_n = b_j$ if $H_n = E_i \cap F_j$ for some $1 \leq i \leq M_1$. □

Lemma 32 (Integral of a Simple Funciton Using Its Disjoint Representation)

Let $E \in \mathfrak{M}(\mathbb{R})$ *and suppose* $\varphi \in \mathrm{SIMP}(E, [0, \infty])$. *If*

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{H_n}$$

is any disjoint representation, then

$$\int_E \varphi = \sum_{n=1}^{n} \alpha_n m H_n.$$

Proof

[3] If $\bigcup_{n=1}^{N} H_n \neq E$, then we set

$$H_{N+1} = E \setminus \bigcup_{n=1}^{N} H_n \text{ and } \alpha_{N+1} = 0.$$

Then

$$\sum_{n=1}^{N} \alpha_n m H_n = \sum_{n=1}^{N+1} \alpha_n m H_n.$$

Thus, wlog, wma

$$\bigcup_{n=1}^{N} H_n = E.$$

Now since the H_n's are mutually disjoint, wma

$$\mathrm{range}\, \varphi = \{\alpha_1, \ldots, \alpha_N\},$$

where we note that the above set may contain repeated elements, i.e. some $\alpha_i = \alpha_j$. We may thus rewrite this set such that

$$\{\alpha_1, \ldots, \alpha_N\} = \{\beta_1 < \beta_2 < \ldots < \beta_M\}$$

and set

$$E_i = \bigcup \{H_j : \alpha_j = \beta_i\}.$$

Note that since $H_i \cap H_j = \emptyset$ for $1 \leq i \neq j \leq N$, for $1 \leq k \leq M$, we have

$$mE_k = \sum_{\alpha_j = \beta_k} m(H_j).$$

[3] One of the problems here is that the disjoint H_n's may not cover the entire domain φ, but we can fill it up with zeros.

Then by definition,

$$\begin{aligned} \int_E \varphi &= \sum_{k=1}^{M} \beta_k \xi_{E_k} \\ &= \sum_{i=1}^{M} \beta_i \sum_{\alpha_j = \beta_i} mH_j \\ &= \sum_{n=1}^{N} \alpha_j mH_j, \end{aligned}$$

as desired. □

Proposition 33 (Linearity and Monotonicity of the Integral of Simple Functions)

Let $E \in \mathfrak{M}(\mathbb{R})$. If $\varphi, \psi \in \mathrm{SIMP}(E, [0, \infty])$ and $\kappa \in [0, \infty)$, then

1. $\int_E \kappa\varphi + \psi = \kappa \int_E \varphi + \int_E \psi$; *and*
2. $\varphi \leq \psi$ *on* E *implies*
$$\int_E \varphi \leq \int_E \psi.$$

Proof

1. By Lemma 31, we can find a common disjoint representation of φ and ψ, say
$$\varphi = \sum_{n=1}^{N} a_n \chi_{H_n} \quad \text{and} \quad \psi = \sum_{n=1}^{N} b_n \chi_{H_n},$$
where the H_n's are pairwise disjoint. Then
$$\kappa\varphi + \psi = \sum_{n=1}^{N} (\kappa a_n + b_n) \chi_{H_n}.$$
Thus by Lemma 32,
$$\begin{aligned} \int_E (\kappa\varphi + \psi) &= \sum_{n=1}^{N} (\kappa a_n + b_n) mH_n \\ &= \kappa \sum_{n=1}^{N} a_n mH_n * \sum_{n=1}^{N} b_n mH_n \end{aligned}$$

$$= \kappa \int_E \varphi + \int_E \psi.$$

2. Using the disjoint representation, if $\varphi \leq \psi$, then $a_n \leq b_n$ for all $1 \leq n \leq N$, and so by Lemma 32,

$$\int_E \varphi = \sum_{n=1}^{N} a_n m H_n \leq \sum_{n=1}^{N} b_n m H_n = \psi.$$ □

We are now ready to define the Lebesgue integral for arbitrary measurable functions.

Definition 29 (Lebesgue Integral)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $f \in \mathcal{L}(E, [0, \infty])$. We define the **Lebesgue integral** *of f as*

$$\int_E^{NEW} f = \sup \left\{ \int_E \varphi : \varphi \in \text{SIMP}(e, [0, \infty)),\, 0 \leq \varphi \leq f \right\}.$$

Note 7.1.2

- *We can actually allow $\varphi \in \text{SIMP}(E, [0, \infty])$.*
- *We put "NEW" in the above integral because we now have "two" definitions for the integral of $\varphi \in \text{SIMP}(E, [0, \infty])$. Writing $\varphi = \sum_{n=1}^{N} \alpha_n \chi_{H_n}$ in its standard form, by Definition 27,*

  $$\int_E \varphi = \sum_{n=1}^{N} \alpha_n m H_n,$$

  while Definition 29 gives us

  $$\int_E^{NEW} \varphi = \sup \left\{ \int_E \psi : \psi \in \text{SIMP}(E, [0, \infty)),\, 0 \leq \psi \leq \varphi \right\}.$$

Remark 7.1.1

Let us try reconciling these two definitions, which will allow us to drop the *dumb-looking "NEW" notation. First, note that*

$$\varphi \in \{\psi \in \mathrm{SIMP}(E, [0, \infty]) : 0 \leq \psi \leq \varphi\},$$

and so by Definition 29, then

$$\int_E \varphi \leq \int_E^{NEW} \varphi.$$

On the other hand, by Proposition 33, if $\varphi \in \mathrm{SIMP}(E, [0, \infty])$ and $0 \leq \psi \leq \varphi$, we have that

$$\int_E \psi \leq \int_E \varphi,$$

and so

$$\int_E^{NEW} \varphi = \sup\left\{\int_E \psi : \psi \in \mathrm{SIMP}(E, 0, \infty]), \ \psi \leq \varphi\right\} \leq \int_E \varphi.$$

Thus

$$\int_E^{NEW} \varphi = \int_E \varphi.$$

With that we shall drop the "NEW" notation from here on.

Definition 30 (Almost Everywhere (a.e.))

*Let $E \in \mathfrak{M}(\mathbb{R})$. We say that a property (P) holds **almost everywhere (a.e.)** on E if the set*

$$B := \{x \in E : (P) \text{ does not hold }\}$$

has Lebesgue measure zero.

Example 7.1.2

Let $E \in \mathfrak{M}(\mathbb{R})$. Given $f, g \in \mathcal{L}(E, \overline{\mathbb{R}})$, we say that $f = g$ a.e. on E if

$$B := \{x \in E : f(x) \neq g(x)\}$$

has measure zero, i.e. $mB = 0$.

An example of this is

$$\chi_{\mathbb{Q}} = 0 = \chi_C$$

a.e. on $\mathbb{R}$, where C is the Cantor set.

Lemma 34 (Monotonicity of the Lebesgue Integral and Other Lemmas)

Let $E \in \mathfrak{M}(\mathbb{R})$ and let $f, g, h : E \to [0, \infty]$ be functions. Suppose that g and h are measurable.

1. *Suppose further that $E = X \dot\cup Y$, where $X, Y \in \mathfrak{M}(E)$. Set $f_1 := f \restriction_X$ and $f_2 := f \restriction_Y$. Then $f \in \mathcal{L}(E, [0, \infty])$ iff f_1 and f_2 are measurable. When this is the case, then*
$$\int_E f = \int_X f_1 + \int_Y f_2.$$
2. *If $g \leq h$, then*
$$\int_E g \leq \int_E h.$$
3. *If $H \in \mathfrak{M}(E)$, then*
$$\int_H g = \int_E g \cdot \chi_H \leq \int_E g.$$

Exercise 7.1.1

Prove Lemma 34.

Proof

1. f is measurable $\iff$ f_1 and f_2 are measurable ($\implies$) Note that
$$f_1 = f \cdot \chi_X \text{ and } f_2 = f \cdot \chi_Y,$$
and since X, Y are measurable, by Proposition 20, we have that f_1 and f_2 are measurable.

   ($\impliedby$) Suppose f_1 and f_2 are measurable and $X \dot\cup Y$. We have that
$$f = f_1 + f_2.$$

I will spare the details, but it is not difficult to see that $\forall a \in \mathbb{R}$, breaking $(a, \infty]$ into disjoint pieces if necessary, $f^{-1}((a, \infty])$ is measurable, and hence f is indeed measurable.

The integral [4] By Definition 27 and Proposition 33, we have

$$\begin{aligned}
\int_E f &= \sup \left\{ \int_E \varphi : \varphi \in \mathrm{SIMP}(E, [0, \infty]),\ \varphi \leq f \right\} \\
&= \sup \left\{ \int_E \varphi \cdot \chi_X + \varphi \cdot \chi_Y : \varphi \in \mathrm{SIMP}(E, [0, \infty]),\ \varphi \leq f \right\} \\
&= \sup \left\{ \int_X \varphi + \int_Y \varphi : \varphi \in \mathrm{SIMP}(E, [0, \infty]),\ \varphi \leq f \right\} \\
&\leq \sup \left\{ \int_X \varphi : \varphi \in \mathrm{SIMP}(X, [0, \infty]),\ \varphi \leq f_1 \right\} \\
&\qquad + \sup \left\{ \int_Y \psi : \psi \in \mathrm{SIMP}(Y, [0, \infty]),\ \psi \leq f_2 \right\} \\
&= \int_X f_1 + \int_Y f_2.
\end{aligned}$$

[4] This proof is iffy.

On the other hand, since $f_1 = f$ on X and $f_2 = f$ on Y, and X and Y are disjoint,

$$\begin{aligned}
&\int_X f_1 + \int_Y f_2 \\
&= \sup \left\{ \int_X \varphi : \varphi \in \mathrm{SIMP}(X, [0, \infty]),\ \varphi \leq f_1 = f \restriction_X \right\} \\
&\qquad + \sup \left\{ \int_Y \psi : \psi \in \mathrm{SIMP}(Y, [0, \infty]),\ \psi \leq f_2 = f \restriction_Y \right\} \\
&= \sup \left\{ \int_X \varphi + \int_Y \psi : \varphi \in \mathrm{SIMP}(X, [0, \infty]), \right. \\
&\qquad \left. \psi \in \mathrm{SIMP}(Y, [0, \infty]),\ \varphi \leq f \restriction_X,\ \psi \leq f \restriction_Y \right\} \\
&= \sup \left\{ \int_E \varphi \cdot \chi_X + \psi \cdot \chi_Y : \varphi, \psi \in \mathrm{SIMP}(E, [0, \infty]), \right. \\
&\qquad \left. \varphi + \psi \leq f \restriction_X + f \restriction_Y = f \right\} \\
&= \int_E f.
\end{aligned}$$

2. By Proposition 30, there exists sequences $\{\varphi_n\}_n$ and $\{\psi_n\}_n$

such that

$$\lim_{n\to\infty} \varphi_n = g \leq h = \lim_{n\to\infty} \psi_n.$$

In particular,

$$\sup_{n\geq 1} \varphi_n = g \leq h = \sup_{n\geq 1} \psi_n.$$

Since the leftmost and rightmost terms are simple functions, by Proposition 33,

$$\begin{aligned}\int_E g &= \sup\left\{\int_E \varphi : \varphi \in \mathrm{SIMP}(E,[0,\infty]),\ \varphi \leq g\right\} \\ &\leq \sup\left\{\int_E \psi : \psi \in \mathrm{SIMP}(E,[0,\infty]),\ \psi \leq h\right\} \\ &= \int_E h.\end{aligned}$$

3. [5] For the first equality, by Definition 27, we have that

   $$\begin{aligned}\int_H g &= \sup\left\{\int_H \varphi : \varphi \in \mathrm{SIMP}(H,[0,\infty]),\ \varphi \leq g\right\} \\ &= \sup\left\{\int_E \varphi\cdot\chi_H : \varphi \in \mathrm{SIMP}(E,[0,\infty]),\ \varphi \leq g\right\} \\ &= \int_E g\cdot\chi_H.\end{aligned}$$

   Note that we have $g\cdot\chi_H \leq g$, and so by part (2),

   $$\int_E g\cdot\chi_H \leq \int_E g. \qquad \square$$

[5] This is also iffy.

Proposition 35 (Integration over Domains of Measure Zero and Integration of Functions Agreeing Almost Everywhere)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $f, g \in \mathcal{L}(E,[0,\infty])$.

1. *If $mE = 0$, then $\int_E f = 0$.*

2. *If $f = g$ a.e. on E, then $\int_E f = \int_E g$.*

Proof

1. $\forall \varphi \in \mathrm{SIMP}(E, [0, \infty])$ written in its standard form

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{E_n},$$

   by monotonicity,

$$0 \leq \int_E \varphi = \sum_{n=1}^{N} \alpha_n m E_n \leq \sum_{n=1}^{N} \alpha_n m E = 0,$$

   and so

$$\int_E \varphi = 0.$$

   Thus

$$\int_E f = \sup \left\{ \int_E \varphi : \varphi \in \mathrm{SIMP}(E, [0, \infty]),\ \varphi \leq f \right\} = \sup\{0\} = 0.$$

2. Let $B := \{x \in E : f(x) \neq g(x)\}$ so that $mB = 0$. Then by Lemma 34 and part (1), we have

$$\begin{aligned} \int_E f &= \int_{E \setminus B} f + \int_B \\ &= \int_{E \setminus B} f + 0 \\ &= \int_{E \setminus B} g + \int_B g \\ &= \int_E g. \end{aligned}$$

□

We are now in a position to prove the following important theorem, which we shall do so next lecture.

Theorem (The Monotone Convergence Theorem)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $(f_n)_n$ be a sequence in $\mathcal{L}(E, [0, \infty])$ such that $f_n \leq f_{n+1}$ a.e. on E. Suppose further that

$$f : E \to [0, \infty]$$

is a function such that $f(x) = \lim_{n \to \infty} f_n(x)$ a.e. on E. Then $f \in$

$\mathcal{L}(E, [0, \infty])$ *and*

$$\int_E f = \lim_{n \to \infty} \int_E f_n.$$

8 Lecture 8 May 30th 2019

8.1 Lebesgue Integration (Continued)

Theorem 36 (The Monotone Convergence Theorem)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $(f_n)_n$ be a sequence in $\mathcal{L}(E, [0, \infty])$ such that $f_n \leq f_{n+1}$ a.e. on E. Suppose further that

$$f : E \to [0, \infty]$$

is a function such that $f(x) = \lim_{n\to\infty} f_n(x)$ a.e. on E. Then $f \in \mathcal{L}(E, [0, \infty])$ and

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

Strategy

1. *Argue why we can proof for the case where we do not have the "a.e." assumption. There are 2 places here where have an "a.e." assumption:*

   (a) $f_n \leq f_{n+1}$ *on* E; *and*

   (b) $f(x) = \lim_{n\to\infty} f_n(x)$ *a.e. on* E.

2. *Look at where good things happen and bad things happen, and we'll be able to show that f is measurable.*

3. *Having gotten rid of the place where nasty things happen and showing that f is measurable. We will find that we need to show that*

$$\int_H f = \lim_{n\to\infty} \int_H f_n,$$

where H is where our hopes and dreams live in.

4. *One direction is easy, since $f_n < f$ for all n, on H. For the other direction, we look at a simple function $\varphi \leq f$, which is then arbitrary. Then since $\lim_{n\to\infty} f_n = f$ (pointwise), we want to be able to show something along the lines of*
$$\int_H f_n - \int_H \varphi \geq 0.$$

> *Instead of trying to do this over the entire H, we can look at where this happens on H for each n. Since the f_n's are increasing, and φ arbitrarily fixed, $f_n - \varphi$ should give us more and more places where they are positive on H.*

Proof

Step 1 Let
$$Z = \left\{ x \in E : f(x) \neq \lim_{n\to\infty} f_n(x) \right\}.$$
By hypothesis, $mZ = 0$ and $Z \in \mathfrak{M}(E)$.

Now by Lemma 34, $f_n \in \mathcal{L}(E, [0, \infty])$ and so $f_n \restriction_{E \setminus Z} \in \mathcal{L}(E \setminus Z, [0, \infty])$. Since by hypothesis we have $\forall x \in E \setminus Z$,
$$f(x) = \lim_{n\to\infty} f_n(x),$$
$f \restriction_{E \setminus Z} \in \mathcal{L}(E \setminus Z, [0, \infty])$ by Corollary 28.

Step 2 For each $n \geq 1$, let
$$Y_n := \{x \in E : f_n(x) > f_{n+1}(x)\}.$$
Then by hypothesis, $mY_n = 0$ and $Y_n \in \mathfrak{M}(E)$. Let
$$Y = \bigcup_{n=1}^{\infty} Y_n.$$
Then since $\mathfrak{M}(E)$ is a σ-algebra, $Y \in \mathfrak{M}(E)$ and
$$0 \leq mY \leq \sum_{n=1}^{\infty} mY_n = 0 \implies mY = 0.$$

1

[1] Up till here, we have showed that we can, instead, turn our focus on wherever nice things happen, and that f is measurable as desired.

At this point, by Lemma 34,

$$\int_E f = \int_{E \setminus (Y \cup Z)} f + \int_{Y \cup Z} f = \int_{E \setminus (Y \cup Z)} f$$

and for each $n \geq 1$,

$$\int_E f_n = \int_{E \setminus (Y \cup Z)} f_n + \int_{Y \cup Z} f_n = \int_{E \setminus (Y \cup Z)} f_n.$$

Thus, it remains for us to show that

$$\int_{E \setminus (Y \cup Z)} f = \lim_{n \to \infty} \int_{E \setminus (Y \cup Z)} f_n.$$

Step 3 Let $X = Y \cup Z$, which then $X \in \mathfrak{M}(E)$ and

$$0 \leq mX \leq mY + mZ = 0 \implies mX = 0.$$

Let $H = E \setminus X$. Note that we then have $H \in \mathfrak{M}(E)$ and $\forall x \in H$,

$$\forall n \geq 1 \quad f_n(x) \leq f_{n+1}(x) \tag{8.1}$$

and

$$f(x) = \lim_{n \to \infty} f_n(x). \tag{8.2}$$

For notational convenience, let

$$g_n = f_n \restriction_H$$

and

$$g = f \restriction_H .$$

By Equation (8.1) and Equation (8.2), we have that

$$g_1 \leq g_2 \leq \ldots \leq g_n \leq g_{n+1} \leq \ldots \leq g.$$

By Lemma 34, $\forall x \in H$

$$\lim_{n \to \infty} g_n(x) = \sup_{n \geq 1} g_n(x) \leq g(x),$$

and so

$$\lim_{n \to \infty} \int_H g_n = \sup_{n \geq 1} \int_H g_n \leq \int_H g.$$

It remains to show that

$$\int_H g \leq \lim_{n\to\infty} \int_H g_n.$$

If we can show that for any $\varphi \in \mathrm{SIMP}(H, [0, \infty])$, we have

$$\lim_{n\to\infty} \int_H g_n \geq \int_H \varphi,$$

then our proof is done, since it would mean that

$$\int_H f = \int_H g = \lim_{n\to\infty} \int_H g_n = \lim_{n\to\infty} \int_H f_n.$$

Step 4 [2] Let $\varphi \in \mathrm{SIMP}(H, [0, \infty])$ such that $\varphi \leq g$. [3] Let $0 < r < 1$, so that either

- $r\varphi = 0 \leq g$; [4] or
- $r\varphi < g = \lim_{n\to\infty} g_n$.

Then, consider

$$H_k = (g_k - r\varphi)^{-1}[0, \infty].$$

Notice that since g_{k_k} is a sequence of increasing functions, we have [5]

$$H_1 \subseteq H_2 \subseteq H_3 \subseteq \ldots$$

Also, note that

$$H = \bigcup_{k=1}^{\infty} H_k.$$

[2] Here, we do something like a race-check. We know that the g_n's grow to be arbitrarily close to g, and the set $\{\varphi \in SIMP(H, [0, \infty]) : \varphi \leq g\}$ also has elements arbitrarily close to g. It would suffice to show that for every φ, the limit of the integral of the g_n's is greater than the integral of φ.

[3] Note that we require this scaling factor, because we cannot allow $\varphi = g$, for otherwise our increasing sequence of g_n's will never be able to 'catch up' to φ, which is what we want.

[4] In the case where $g = 0$, we have that $r\varphi = 0$ and not something bigger.

[5] The increasing-ness of the g_k's guarantees that if $(g_k - r\varphi)(x) \geq 0$, then $(g_{k+1} - r\varphi)(x) \geq 0$. This is sort of like a rising water level scenario.

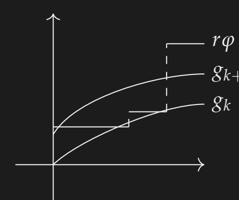

Figure 8.1: Increasing levels of g_k 'covers' more and more parts of $r\varphi$

[6] WTS
$$\int_H \varphi = \lim_{k\to\infty} \int_{H_k} \varphi.$$
Since $\varphi \in \mathrm{SIMP}(H, [0, \infty])$, let us write
$$\varphi = \sum_{k=1}^{N} \alpha_k \chi_{J_k}$$
in its standard form, where $J_k \in \mathfrak{M}(H)$. Then
$$\int_H \varphi = \sum_{k=1}^{N} \alpha_k m J_k,$$
while
$$\int_{H_n} \varphi = \sum_{k=1}^{N} \alpha_k m(J_k \cap H_n)$$
for each $n \geq 1$.

By the **continuity of the Lebesgue measure** (A1), notice that
$$\lim_{n\to\infty} m(J_k \cap H_n) = m\left(J_k \cap \left(\bigcup_{n=1}^{\infty} H_n\right)\right) = m(J_k \cap H) = m(J_k)$$
Thus
$$\lim_{n\to\infty} \int_{H_n} \varphi = \sum_{k=1}^{N} \alpha_k m(J_n) = \int_H \varphi,$$
as claimed.

Then in particular, we have that
$$\int_H r\varphi = \lim_{k\to\infty} \int_{H_k} \varphi \leq \lim_{k\to\infty} \int_{H_k} g_k \leq \lim_{k\to\infty} \int_H g_k,$$
where the last inequality follows from Lemma 34.

This is exactly the final piece that we have set out to prove, and so we have completed the proof. □

[6] By this construction, we have that $r\varphi \leq g_k$ in H_k for each k. So we already have
$$\lim_{k\to\infty} \int_{H_k} r\varphi \leq \lim_{k\to\infty} \int_{H_k} g_k$$
in our bag. Notice that since φ is a simple function, by Definition 27, we have
$$\int_{H_k} \varphi = \int_H \varphi \cdot \chi_{H_k}.$$
Since the H_k's is an 'increasing sequence' of sets, and especially since $H = \bigcup_{k=1}^{\infty} H_k$, we expect
$$\lim_{k\to\infty} \int_{H_k} \varphi = \int_H \varphi.$$

Example 8.1.1

Recall our "pathological" sequence of Riemann integral functions earlier on, where $E = \mathbb{Q} \cap [0, 1] = \{q_n\}_{n=1}^{\infty}$, and sequence of functions
$$f_n = \chi_{\{q_1,\ldots,q_n\}}, \text{ for } n \geq 1,$$

and their limit

$$f = c_{\mathbb{Q}\cap[0,1]}.$$

We have that

$$0 \leq f_1 \leq f_2 \leq \ldots \leq f,$$

and each [7]

$$f_n \in \mathcal{L}([0,1], [0, \infty)).$$

[7] **Exercise 8.1.1**
Show that $f_n \in \mathcal{L}([0,1], [0, \infty))$.

By the Monotone Convergence Theorem (MCT), f is measurable and

$$\int_{[0,1]} f = \lim_{n\to\infty} \int_{[0,1]} f_n = \lim_{n\to\infty} 0 = 0.$$

This agrees with what we saw much earlier on, i.e.

$$0 \leq \int_{[0,1]} f = \int_{[0,1]} \chi_E = mE \leq m\mathbb{Q} = 0.$$

Note that f_n is Riemann integrable, but f is not, but it is Lebesgue integrable. In other words, this function f is an example of a Lebesgue integrable function that is not Riemann integrable.

Corollary 37 (Linearity of the Lebesgue Integral and Other Results)

Let $E \in \mathfrak{M}(\mathbb{R})$.

1. *If $f, g \in \mathcal{L}(E, [0, \infty])$ and $\kappa \geq 0$, then*

$$\int_E \kappa f + g = \kappa \int_E f + \int_E g.$$

2. *If $(h_n)_{n=1}^\infty$ is a sequence in $\mathcal{L}(E, [0, \infty])$ and*

$$h(x) := \lim_{N\to\infty} \sum_{n=1}^N h_n(x), \quad \forall x \in E,$$

then $h \in \mathcal{L}(E, [0, \infty])$ and

$$\int_E h = \sum_{n=1}^\infty \int_E h_n.$$

3. *Let $f \in \mathcal{L}(E, [0, \infty])$. If $(H_n)_{n=1}^\infty$ is a sequence $\mathfrak{M}(E)$ with $H_i \cap H_j =$*

$\emptyset$ *when* $1 \leq i \neq j \leq \infty$ *and* $H = \dot{\bigcup}_{n=1}^{\infty} H_n$, *then*

$$\int_H f = \sum_{n=1}^{\infty} \int_{H_n} f.$$

Proof

1. By A3, there exists a sequence of simple, measurable functions $(\varphi_n)_n, (\psi_n)_n$ in $\mathcal{L}(E, [0, \infty])$ such that

$$0 \leq \varphi_1 \leq \varphi_2 \leq \ldots \leq f$$
$$0 \leq \psi_1 \leq \psi_2 \leq \ldots \leq g$$

such that $\forall x \in E$,

$$\lim_{n \to \infty} \varphi_n(x) = f(x)$$
$$\lim_{n \to \infty} \psi_n(x) = g(x)$$

By Proposition 33, we have that for each n, for any $\kappa \in E$, we have

$$\int_E \kappa\varphi_n + \psi_n = \kappa \int_E \varphi_n + \int_E \psi_n.$$

Furthermore, note that

$$\lim_{n \to \infty} (\kappa\varphi + \psi)(x) = (\kappa f + g)(x),$$

and $(\kappa\varphi_n + \psi_n)_n$ is an increasing [8] sequence of non-negative, simple, measurable functions converging pointwise to the function $\kappa f + g$.

[8] If you are second-guessing yourself like I did, notice that that n is fixed for both of them, not just one of them.

Thus, by the MCT, we see that

$$\begin{aligned}\int_E (\kappa f + g) &= \lim_{N \to \infty} (\kappa\varphi_N + \psi_N) \\ &= \lim_{N \to \infty} \kappa \int_E \varphi_N + \int_E \psi_N = \kappa \int_E f + \int_E g.\end{aligned}$$

2. [9] Let

$$g_N = \sum_{n=1}^{N} h_n$$

for each $N \geq 1$.

[9] Since range $h_n \subseteq [0, \infty]$, the partial sums form an increasing sequence of functions. Then, we can make use of the MCT.

Showing that $g_N \in \mathcal{L}(E, [0, \infty])$ Let $(a, \infty]$, for any $\alpha \in [0, \infty)$. Then since g_N is a finite sum of functions, we have that

$$g_N((a,\infty]) = h_1((a,\infty]) \cup h_2((a,\infty]) \cup \ldots \cup h_N((a,\infty]),$$

which is a countable union of measurable sets, and is hence measurable.

Then

$$0 \leq g_1 \leq g_2 \leq \ldots \leq h$$

and $\forall x \in E$

$$\lim_{N\to\infty} g_N(x) = h(x),$$

both of which are from our assumptions.

By the MCT and part (1), we have that

$$\begin{aligned} \int_E h &= \lim_{N\to\infty} \int_E g_N \\ &= \lim_{N\to\infty} \int_E \sum_{n=1}^N h_n \\ &= \lim_{N\to\infty} \sum_{n=1}^N \int_E h_n \\ &= \sum_{n=1}^\infty \int_E h_n \end{aligned}$$

as required.

3. [10] Let

$$h_n = f \cdot \chi_{H_n}$$

for each $n \geq 1$. Since each $H_n \in \mathfrak{M}(E)$, each χ_{H_n}, and f being measurable implies that each h_n is measurable. Since $H_i \cap H_j = \emptyset$ for all $1 \leq i \neq j \leq \infty$, we have that

$$f = \sum_{n=1}^\infty h_n.$$

By part (2), we have that

$$\int_E f = \sum_{n=1}^\infty \int_E h_n = \sum_{n=1}^\infty \int_E f \cdot \chi_{H_n} = \sum_{n=1}^\infty \int_{H_n} f. \qquad \square$$

[10] Since the RHS of the goal integrates over $H_n \subseteq H$, and the H_i's are disjoint, we can break f down by where H_n is defined.

Definition 31 (Lebesgue Integrable)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $f \in \mathcal{L}(E, \overline{\mathbb{R}})$. We say that f is ***Lebesgue integrable*** *on E if*[11]

$$\int_E f^+ < \infty \quad \text{and} \quad \int_E f^- < \infty,$$

in which case we set

$$\int_E f := \int_E f^+ - \int_E f^-.$$

We denote by $\mathcal{L}_1(E, \overline{\mathbb{R}})$ the set of all Lebesgue integrable functions from E to $\overline{\mathbb{R}}$, and $\mathcal{L}_1(E, \mathbb{R})$ all Lebesgue integrable functions from E to $\mathbb{R}$.

[11] Recall Remark 6.1.1.

Remark 8.1.1

Let $E \in \mathfrak{M}(\mathbb{R})$.

1. *By definition, every Lebesgue integrable function on E is Lebesgue measurable.*

2. *A* ***measurable function*** *f is Lebesgue integrable iff $|f|$ is Lebesgue integrable. Notice that*

$$\int_E f = \int_E f^+ - \int_E f^-$$

while

$$\int_E |f| = \int_E f^+ + f^- = \int_E f^+ + \int_E f^-,$$

and so if either of these are integrable, then

$$\int_E f^+ < \infty \quad \text{and} \quad \int_E f^- < \infty,$$

which then the other must also be integrable.

It is important to note that this is a distinguishing feature of Lebesgue integration, in comparison to Riemann integration. For instance, if we consider the function

$$f(x) = \frac{\sin x}{x}, \text{ for } x \geq 1,$$

improper Riemann integration gives us that $\int_1^\infty f(x)\,dx = \frac{\pi}{2}$. But from *the POV of Lebesgue integration, notice that*

$$\begin{aligned}
&\int_{[\pi,(N+1)\pi]} \left| \frac{(\sin x)^+}{x} \right| \\
&= \sum_{k=1}^{N} \int_{[\pi k, \pi(k+1)]} \left| \frac{(\sin x)^+}{x} \right| \\
&= \sum_{k=1}^{N} \int_{[0,\pi]} \frac{|\sin(t + k\pi)|}{t + k\pi} \\
&= \sum_{k=1}^{N} \int_{[0,\pi]} \frac{|\sin t|}{t + k\pi} \\
&\geq \sum_{k=1}^{N} \frac{1}{(k+1)\pi} \int_{[0,\pi]} \sin t.
\end{aligned}$$

Assuming we know some of the upcoming results, in particular, assuming that we know that for bounded functions the Lebesgue integral is the same as the Riemann integral, we see that the above is

$$= \frac{2}{\pi} \sum_{k=1}^{N} \frac{1}{k+1},$$

which is a harmonic series and hence divergent.

3. *If* $f \in \mathcal{L}_1(E, \overline{\mathbb{R}})$*, then*

$$mf^{-1}(\{-\infty\}) = 0 = mf^{-1}(\{\infty\}).$$

Exercise 8.1.2

Prove that the above is indeed the case.

4. *Following the above, if we set*

$$H = f^{-1}(\{-\infty, \infty\}),$$

then $H \in \mathfrak{M}(E)$ *and* $mH = 0$*. Letting*

$$g = f \cdot \chi_{E \setminus H}.$$

Then

$$g = f \text{ a.e. and } g \in \mathcal{L}_1(E, \mathbb{R}).$$

*This will prove itself more useful than it seems, especially since $\mathcal{L}_1(E, \overline{\mathbb{R}})$ is that it is **not a vector space**!!!*

5. *Suppose that $g : E \to \mathbb{C}$ is measurable. Let us write*

$$g = (g_1 - g_2) + i(g_3 - g_4),$$

*where $g_1 = (\Re g)^+$, $g_2 = (\Re g)^-$, $g_3 = (\Im g)^+$ and $g_4 = (\Im g)^-$. Then we say that g is **Lebesgue integrable**, and write*

$$g \in \mathcal{L}_1(E, \mathbb{C}),$$

if

$$\int_E g_k < \infty \quad \forall 1 \leq k \leq 4,$$

and we write

$$\int_E g = \left(\int_E g_1 - \int_E g_2 \right) + i \left(\int_E g_3 + \int_E g_4 \right).$$

Proposition 38 (Linearity of Lebesgue Integral for Lebesgue Integrable Functions)

let $E \in \mathfrak{M}(\mathbb{R})$. Suppose that $f, g \in \mathcal{L}_1(E, \mathbb{R})$ and $\kappa \in \mathbb{R}$.

1. *$\kappa f \in \mathcal{L}_1(E, \mathbb{R})$ and $\int_E \kappa f = \kappa \int_E f$.*

2. *$f + g \in \mathcal{L}_1(E, \mathbb{R})$ and $\int_E (f + g) = \int_E f + \int_E g$.*

3. *Finally,*

$$\left| \int_E f \right| \leq \int_E |f|.$$

Proof

Note that Corollary 37 covers for the cases where $f, g \in \mathcal{L}(E, [0, \infty])$ and $\kappa \geq 0$ for (1) and (2). This is, unfortunately, insufficient for the entire proposition

1. Let $f = f^+ - f^-$.

Case 1: $\kappa = 0$ We have that

$$\int_E \kappa f = \int_E 0 = 0 = \kappa \int_E f.$$

Case 2: $k > 0$ We have

$$\kappa f = (\kappa f)^+ - (\kappa f)^-.$$

Note

$$(\kappa f)^+ = \kappa f^+ \text{ and } (\kappa f)^- = \kappa f^-.$$

So, since $f^+, -f^- \in \mathcal{L}(E, [0, \infty])$, by Corollary 37,

$$\begin{aligned}
\int_E \kappa f &= \int_E \kappa f^+ - \int_E \kappa f^- \\
&= \kappa \int_E f^+ - \kappa \int_E f^- \\
&= \kappa \left(\int_E f^+ - f^- \right) \\
&= \kappa \int_E f.
\end{aligned}$$

Case 3: $\kappa < 0$ Similar to the above, we first observe that

$$(\kappa f)^+ = -\kappa f^- \text{ and } (\kappa f)^- = -\kappa f^+.$$

Then by the same reason as in the last case, we have

$$\begin{aligned}
\int_E \kappa f &= \int_E -\kappa f^- - \int_E -\kappa f^+ \\
&= -\kappa \left(\int_E f^- - \int_E f^+ \right) \\
&= -\kappa \left(- \int_E f \right) \\
&= \kappa \int_E f.
\end{aligned}$$

2. $f + g \in \mathcal{L}_1(E, \mathbb{R})$ For convenience, let

$$h = f + g = f^+ - f^- + g^+ - g^-.$$

Notice that

$$h^+, h^- \leq |h| = |f + g| \leq |f| + |g| = f^+ + f^- + g^+ + g^-.$$

Thus by ✈Corollary 37,

$$\int_E h^+ \leq \int_E f^+ + f^- + g^+ + g^- \\ = \int_E f^+ + \int_E f^- + \int_E g^+ + \int_E g^- < \infty.$$

Similarly, $\int_E h^- < \infty$.

$\int_E (f+g) = \int_E f + \int_E g$ Notice that

$$h^+ - h^- = h = f + g = f^+ - f^- + g^+ - g^-,$$

and so

$$h^+ + f^- + g^- = h^- + f^+ + g^+.$$

Then by ✈Corollary 37,

$$\int_E h^+ + \int_E f^- + \int_E g^- = \int_E h^- + \int_E f^+ + \int_E g^+,$$

and so

$$\int_E (f+g) = \int_E h = \int_E h^+ - \int_E h^- \\ = \int_E f^+ - \int_E f^- + \int_E g^+ - \int_E g^- \\ = \int_E f + \int_E g.$$

3. First, notice that $f \in \mathcal{L}_1(E, \mathbb{R})$, by our previous remark, $|f| \in \mathcal{L}_1(E, \mathbb{R})$. Now since $|f| = f^+ + f^-$, and $\left|\int_E f^+\right|, \left|\int_E f^-\right| \geq 0$,

$$\left|\int_E f\right| = \left|\int_E f^+ - \int_E f^-\right| \\ \leq \left|\int_E f^+\right| + \left|\int_E f^-\right| \\ = \int_E f^+ + \int_E f^- \\ = \int_E |f|.$$

□

9 Lecture 9 Jun 04 2019

9.1 Lebesgue Integration (Continued 2)

Thus far, we've only integrated simple functions, and never even did so for, say, $f(x) = x$. Trying to do that will lead to intense swearing, rising of blood pressure, heavy signs of nausea and mental pain. Why? Well just try doing it.

Exercise 9.1.1 (How a slime became one heck of a monster to deal with)

Calculate $\int_{[0,1]} x$.

We hate pain, and now we want to crawl back to Riemann integration and ask for forgiveness. Fortunately, the nice world of Riemann integration is kind enough to give us a bridge. We shall now study this bridge. In particular, we shall see that for **bounded** functions on **closed, bounded** intervals, Riemann integrability implies Lebesgue integrability, and, in fact, they coincide on these functions. In particular, this opens up the **Fundamental Theorem of Calculus** (for Riemann integration) to us.

Lemma 39 (Riemann Integrability and Lebesgue Integrability of Step Functions)

Let $a < b \in \mathbb{R}$ *and* $\varphi : [a, b] \to \mathbb{R}$ *be a step function. Then* φ *is both Riemann integrable and Lebesgue integrable, and*

$$\int_{[a,b]} \varphi = \int_a^b \varphi.$$

Proof

Let $P = \{a = p_0 < p_1 < p_2 < \ldots p_N = b\} \in \mathcal{P}([a,b])$, where the p_n's are chosen such that $[p_{n-1}, p_n)$ do not contain a 'jump'. Since φ is a step function, let

$$\varphi = \sum_{n=1}^{N} \alpha_n \chi_{[p_{n-1}, p_n)},$$

where $\alpha_n = \varphi(x)$ for all $x \in [p_{n-1}, p_n)$, $1 \leq n \leq N$.

Then

$$\begin{aligned}
\int_{[a,b]} \varphi &= \sum_{n=1}^{N} \alpha_n m[p_{n-1}, p_n) \\
&= \sum_{n=1}^{N} \alpha_n (p_n - p_{n-1}) \\
&= \sum_{n=1}^{N} \int_{p_{n-1}}^{p_n} \alpha_n \\
&= \int_a^b \sum_{n=1}^{N} \alpha_n \chi_{[} p_{n-1}, p_n) \\
&= \int_a^b \varphi.
\end{aligned}$$

□

Theorem 40 (Bounded Riemann-Integrable Functions are Lebesgue Integrable)

Let $a < b \in \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ be a bounded, Riemann-integrable function. Then $f \in \mathcal{L}_1([a,b], \mathbb{R})$ and

$$\int_{[a,b]} f = \int_a^b f,$$

i.e. the Lebesgue and Riemann integrals of f over $[a,b]$ coincide.

Strategy

Here is my understanding of the idea that motivates this proof.

1. *It is important that the function is bounded both on its domain and its range. A bound on the domain allows us to do finite sums, and a bound on the range puts a cap on how high our rectangles can be.*

2. *We need to reduce the problem to deal only with step functions, using step functions as close to f as possible, and then use our earlier results and intuition to forge forward.*

Proof

First, since f is bounded, wma $|f| < M \in \mathbb{R}$. Let $g = M\chi_{[a,b]}$, which is a step-function and is hence integrable by Lemma 39. Then, notice that $f + g$ is Riemann integrable. Furthermore, observe that

$$\int_a^b (f+g) = \int_a^b f + M(b-a).$$

So $f + g \in \mathcal{L}_1([a,b], \mathbb{R})$ iff $f \in \mathcal{L}_1([a,b], \mathbb{R})$.

Now, by Theorem 2, for each $n \geq 1$, $\exists R_n \in \mathcal{P}[a,b]$ partition such that $\forall X, Y \supseteq R_n$ refinements, $\forall X^*, Y^*$ test values of X and Y respectively, we have

$$|S(f, X, X^*) - S(f, Y, Y^*)| < \frac{1}{N}.$$

[1] Now, let $Q_N = \bigcup_{n=1}^{N} R_n$, so that it is a **common refinement** of $R_1, R_2, \ldots, R_N$. Write

$$Q_N = \left\{ a = q_{0,N} < q_{1,N} < \ldots < q_{m_N,N} \right\}.$$

[2] Let

$$H_{k,N} = [q_{k,N}, q_{k+1,N}] \text{ for } 1 \leq k \leq m_N - 1,$$

and

$$H_{m_N,N} = [q_{m_N-1,N}, q_{m_N,N}].$$

[3] Define for each $1 \leq k \leq m_N$,

$$\alpha_{k,N} := \inf\{f(t) : t \in H_{k,N}\} \leq -M$$
$$\beta_{k,N} := \sup\{f(t) : t \in H_{k,N}\} \leq M.$$

[1] Get finer and finer refinements.

[2] Look at each subinterval of each refinement.

[3] Get the sup and inf of each interval under f.

[4] For each $N \geq 1$, let

$$\varphi_N := \sum_{k=1}^{m_N} \alpha_{k,N} \chi_{H_{k,N}}$$
$$\psi_N = \sum_{k=1}^{m_N} \beta_{k,N} \chi_{H_{k,N}}.$$

[4] Use the above α's and β's to construct simple functions, which are step-like functions.

Since each φ_N, ψ_N is simple, they are all measurable and Lebesgue integrable (cf. Lemma 39).

Now, notice that

$$Q_1 \subseteq Q_2 \subseteq \ldots \subseteq Q_N \subseteq Q_{N+1} \subseteq \ldots$$

since it is a sequence of finer and finer refinements, we have

$$\varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \ldots \leq f \leq \ldots \leq \psi_3 \leq \psi_2 \leq \psi_1. \tag{9.1}$$

Thus, by Lemma 39 and Lemma 34, we have

$$\int_{[a,b]} \varphi_N = \int_a^b \varphi_N \leq \int_a^b f \leq \int_a^b \psi_N = \int_{[a,b]} \psi_N$$

for each N. Since Q_N is a refinement of R_N, we have that

$$|S(f, Q_N, Q_N^*) - S(f, Q_N, Q_N^{**})| < \frac{1}{N},$$

which implies

$$\left| \int_{[a,b]} \varphi_N - \int_{[a,b]} \psi_N \right| < \frac{1}{N},$$

for $N \geq 1$.

Due to Equation (9.1), let

$$\varphi := \lim_{N \geq 1} \varphi_N \text{ and } \psi := \lim_{N \geq 1} \psi_N.$$

Then by the MCT, we have that

$$\begin{aligned}
\int_{[a,b]} \varphi &= \lim_{N \to \infty} \int_{[a,b]} \varphi_N = \lim_{N \to \infty} \int_a^b \varphi_N \\
&= \int_a^b f \\
&= \lim_{N \to \infty} \int_a^b \psi_N = \lim_{N \to \infty} \psi_N = \int_{[a,b]} \psi.
\end{aligned}$$

Then $\int_{[a,b]} \varphi - \psi = 0$. Since $\varphi \leq \psi$, we must thus have $\varphi = \psi$ a.e. on $[a,b]$. Since $\varphi \leq f \leq \psi$, we have that $\varphi = f = \psi$ a.e. on $[a,b]$. Since φ, ψ are measurable, so is f, and thus

$$\int_{[a,b]} f = \int_{[a,b]} \varphi = \int_a^b f < \infty.$$

□

Corollary 41 (Bounded Riemann-Integrable Functions are Lebesgue Integrable – Complex Version)

Let $a < b \in \mathbb{R}$ and $f : [a,b] \to \mathbb{C}$ be a bounded, Riemann-integrable function. Then $g \in \mathcal{L}_1([a,b], \mathbb{C})$ and

$$\int_{[a,b]} f = \int_a^b f.$$

Our earlier demon-level slime has been reduced back to being a, well, slime-level monster.

Example 9.1.1

Let $f(x) = x$ and $x \in [0,1]$. Then by the **Fundamental Theorem of Calculus**,

$$\int_{[0,1]} f = \int_0^1 f = \frac{x^2}{2}\bigg|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

Example 9.1.2

Let $f(x) = \frac{1}{x^2}$ where $x \in E := [1, \infty)$. We want to calculate $\int_{[1,\infty)} f$. For each $n \geq 1$, set $f_n := f \cdot \chi_{[1,n]}$. Then f is measurable, since it is continuous except at one point on E, and

$$0 \leq f_1 \leq f_2 \leq \ldots,$$

with

$$\lim_{n \to \infty} f_n(x) = f(x) \quad \forall x \geq 1.$$

By Theorem 40, for all $n \geq 1$,

$$\int_{[1,n]} f_n = \int_1^n f_n = \int_1^n \frac{1}{x^n} = -\frac{1}{x}\bigg|_1^n = 1 - \frac{1}{n}.$$

By the MVT,

$$\begin{aligned}\int_{[1,\infty)} f &= \lim_{n\to\infty} \int_{[1,n]} f_n \\ &= \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1.\end{aligned}$$

Note 9.1.1

In the above example, the Lebesgue integral of f returns the value of the improper Riemann integral of f over $[1,\infty)$, which is not what happened in another function that we looked at earlier. There are 2 things to note here:

- *it is possible for an improper Riemann integral of a measurable function $f : [1,\infty) \to \mathbb{R}$ to exist, even though the Lebesgue integral $\int_{[1,\infty)} f$* ***does not exist****!*
- *There is no notion of an "improper" Lebesgue integral. The domain of f, $[1,\infty)$, is just another measurable set.*

In the Monotone Convergence Theorem, if the "increasing" assumption is dropped, then the result may not hold.

Example 9.1.3 (The MCT needs an increasing/decreasing sequence of functions)

Consider the sequence $(f_n)_n$ given by

$$f_n : [1,\infty) \to \mathbb{R}$$

where

$$x \mapsto \begin{cases} \frac{1}{nx} & 1 \leq x \leq e^n \\ 0 & x > e^n \end{cases}.$$

Then $(f_n)_n$ converges **uniformly** to $f = 0$ on $[1,\infty)$. Note that for all $n \geq 1$, f_n is Riemann integrable, and bounded on $[1, e^n]$, and so

$$\begin{aligned}\int_{[1,\infty)} f_n &= \int_{[1,e^n]} \frac{1}{nx} \\ &= \int_1^{e^n} \frac{1}{nx}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n} \ln x \Big|_1^{e^n} \\
&= \frac{1}{n}(n - 0) = 1,
\end{aligned}
$$

for each $n \geq 1$. However,

$$\int_{[1,\infty)} f = \int_{[1,\infty)} f = 0.$$

10 Lecture 10 Jun 06th 2019

10.1 Lebesgue Integration (Continued 3)

Theorem 42 (Fatou's Lemma)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $f_n \in \mathcal{L}(E, [0, \infty])$, for $n \geq 1$. Then

$$\int_E \liminf_{n \geq 1} f_n \leq \liminf_{n \geq 1} \int_E f_n.$$

Proof

For each $N \geq 1$, set $g_N = \inf\{f_n : n \geq N\}$. Then by Proposition 27, each g_N is measurable, and clearly

$$g_1 \leq g_2 \leq g_3 \leq \ldots.$$

Then by the MCT, we have

$$\int_E \liminf_{n \geq 1} f_n = \int_E \lim_{N \to \infty} g_N = \lim_{N \to \infty} \int_E g_N.$$

Since $g_N \leq f_n$ for all $n \geq N$ (by construction), we have

$$\int_E g_N \leq \int_E f_n$$

for all $n \geq N$, whence

$$\int_E g_N \leq \liminf_{n \geq 1} \int_E f_n.$$

Since this holds for all $N \geq 1$, we have that

$$\int_E \liminf_{n\geq 1} f_n = \lim_{N\to\infty} \int_E g_N \leq \liminf_{n\geq 1} \int_E f_n. \qquad \square$$

An example where the inequality in Fatou's Lemma is strict is the following.

Example 10.1.1

Consider a sequence of functions $f_n = n\chi_{\left(0,\frac{1}{n}\right]}$, $n \geq 1$. It's clear that for any $x \in [0,1]$, $\lim_{n\to\infty} f_n(x) = 0$. Thus

$$\int_{[0,1]} \liminf_{n\geq 1} f_n = \int_{[0,1]} 0 = 0.$$

On the other hand

$$\int_{[0,1]} f_n = nm\left(\left(0, \frac{1}{n}\right]\right) = 1$$

for all $n \geq 1$, and so $\liminf_{n\geq 1} \int_{[0,1]} f_n = 1$.

Example 10.1.2

Suppose $E \in \mathfrak{M}(\mathbb{R})$, $f \in \mathcal{L}(E, \overline{\mathbb{R}})$. Recall that $f \in \mathcal{L}_1(E, \overline{\mathbb{R}}) \iff |f| \in \mathcal{L}_1(E, \overline{\mathbb{R}})$.

Suppose $g \in \mathcal{L}_1(E, \overline{\mathbb{R}})$, $f \in \mathcal{L}(E, \overline{\mathbb{R}})$ and suppose $0 \leq |f| \leq g$ a.e. on E, and that $\int_E g < \infty$. Then $\int_E |f| \leq \int_E g < \infty$, which thus $f \in \mathcal{L}_1(E, \overline{\mathbb{R}})$.

Theorem 43 (Lebesgue Dominated Convergence Theorem)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $(f_n)_n$ in $\mathcal{L}_1(E, \overline{\mathbb{R}})$. Suppose that there exists $g \in \mathcal{L}_1(E.\overline{\mathbb{R}})$ such that $|f_n| \leq g$ a.e. on E, for $n \geq 1$. Suppose furthermore that $f : E \to \overline{\mathbb{R}}$ is a function, and that

$$f(x) = \lim_{n\to\infty} f_n(x), \text{ a.e. on } E.$$

Then $f \in \mathcal{L}_1(E, \overline{\mathbb{R}})$ *and*

$$\int_E f = \lim_{n \to \infty} \int_E f_n.$$

Proof

Isolating "bad" points Consider, for each $n \geq 1$, the set

$$Y_n := \{x \in E : |f_n(x)| > g(x)\}.$$

By assumption, $mY_n = 0$ for each $n \geq 1$. Letting

$$Y := \bigcup_{n=1}^{\infty} Y_n = \{x \in E : |f_n(x)| > g(x), n \geq 1\},$$

we have that

$$0 \leq mY \leq \sum_{n=1}^{\infty} mY_n = 0,$$

and so $mY = 0$.

Furthermore, consider

$$Z := \{x \in E : f(x) \neq \lim_{n \to \infty} f_n(x)\}.$$

By assumption, $mZ = 0$.

Let

$$B := Y \cup Z.$$

Then $\forall x \in B$, we have

$$f(x) \neq \lim_{n \to \infty} f_n(x) \text{ and } |f_n(x)| > g(x) \text{ for each } n \geq 1.$$

Most importantly, we have that

$$0 \leq mB \leq mY + mZ = 0,$$

and so $mB = 0$.

Let $H = E \setminus B$. Then $\forall x \in H$,

$$f(x) = \lim_{n \to \infty} f_n(x) \text{ and } |f_n(x)| \leq g(x) \text{ for each } n \geq 1.$$

It follows that if we can prove the statement for

$$f_n \restriction_H \text{ and } f \restriction_H,$$

then we obtain the result that we desire. Thus, wlog, we may replace E with H.

Proving the statement Since $f(x) = \lim_{n\to\infty} f_n(x)$, by A2, we have that

$$\limsup_{n\geq 1} f_n(x) = \liminf_{n\geq 1} f_n(x) = \lim_{n\to\infty} f_n(x) = f(x),$$

and so in particular we have

$$\int_E f = \int_E \liminf_{n\geq 1} f_n(x).$$

From Fatou's Lemma and A2, we have that

$$\int_E f \leq \liminf_{n\geq 1} \int_E f_n \leq \limsup_{n\geq 1} \int_E f_n.$$

Now, notice that $g - f_n \geq 0$ [1], and we have

[1] This is required to invoke Fatou's Lemma

$$\begin{aligned} \int_E (g-f) &= \int_E (g - \limsup_{n\geq 1} f_n) \\ &= \int_E \liminf_{n\geq 1} (g - f_n) \\ &\leq \liminf_{n\geq 1} \int_E (g-f_n) \quad \because \textit{Fatou's} \\ &= \int_E g - \limsup_{n\geq 1} \int_E f_n. \end{aligned}$$

Thus

$$\int_E f \geq \limsup_{n\geq 1} \int_E f_n.$$

Therefore

$$\limsup_{n\geq 1} \int_E f_n \leq \int_E f \leq \liminf_{n\geq 1} \int_E f_n \leq \limsup_{n\geq 1} \int_E f_n.$$

By the **Squeeze Theorem**, we obtain

$$\int_E f = \lim_{n\to\infty} \int_E f_n. \qquad \square$$

10.2 L_p Spaces

Functional analysis is the study of normed linear spaces and the continuous linear maps between them. Amongst the most important examples are the so-called L_p-spaces, and we will now turn our attention towards them. .

You may wish to refresh your memory on the definition of a semi-norm.

Example 10.2.1

Let $E \in \mathfrak{M}(\mathbb{K})$ and $mE > 0$. Recall that

$$\mathcal{L}_1(E, \mathbb{K}) = \left\{ f \in \mathcal{L}(E, \mathbb{K}) : \int_E |f| < \infty \right\}.$$

Define the map

$$\begin{aligned} \nu_1 : \mathcal{L}_1(E, \mathbb{K}) &\to \mathbb{K} \\ f &\mapsto \int_E |f| . \end{aligned}$$

Observe that

- $\nu_1(f) \geq 0$ for all $f \in \mathcal{L}_1(E, \mathbb{K})$;
- $\nu_1(0) = \int_E |0| = 0$;
- $\kappa \in \mathbb{K} \implies$

  $$\nu_1(\kappa f) = \int_E |\kappa f| = |\kappa| \int_E |f| = |\kappa| \, \nu_1(f);$$

  and
- $\forall f, g \in \mathcal{L}_1(E, \mathbb{K})$

  $$\nu_1(f+g) = \int_E |f+g| \leq \int_E |f| + \int_E |g| = \nu_1(f) + \nu_1(g).$$

However, it is important to notice that for any $x_0 \in E$,

$$\nu_1(\chi_{\{x_0\}}) = \int_{\{x_0\}} 1 = 0.$$

Thus ν_1 is **not a norm** since $\chi_{\{x_0\}} \neq \emptyset$.

Proposition 44 (Kernel of a Vector Space is a Linear Manifold)

Let $\mathcal{W}$ be a vector space over the field $\mathbb{K}$, and suppose that ν is a seminorm on $\mathcal{W}$. Let

$$\mathcal{N} := \{w \in W : \nu(w) = 0\}.$$

Then $\mathcal{N}$ is a ***linear manifold*** [2] *in $\mathcal{W}$ and so $\mathcal{W}/\mathcal{N}$ is a vector space over $\mathbb{K}$, whose elements we denote by*

$$[x] := x + \mathcal{N}.$$

Furthermore, the map

$$\begin{aligned} \|\cdot\| : \mathcal{W}/\mathcal{N} &\to \mathbb{K} \\ [x] &\mapsto \nu(x) \end{aligned}$$

is well-defined, and defines a norm on $\mathcal{W}/\mathcal{N}$.

[2] A subspace M of a Hilbert space, which is a vector space with an inner product such that its **induced** norm, which in turn induces a metric on the space, makes the space a complete metric space, is called a linear manifold if it is **closed under addition and scalar multiplication**. (Source: Stover (nd))
Here, we can safely talk about Hilbert spaces because $\mathbb{K}$ is endowed with an inner product. Furthermore, the check is to simply show that M is a subspace of the original space.

Proof

$\mathcal{N}$ is a linear manifold Firstly, note that $\nu(0) = 0 \implies 0 \in \mathcal{N}$. Thus $\mathcal{N} \neq \emptyset$. Let $x, y \in \mathcal{N}$ and $\kappa \in \mathbb{K}$. Then

$$0 \leq \nu(\kappa x + y) \leq |\kappa| \, \nu(x) + \nu(y) = 0,$$

which implies

$$\nu(\kappa x + y) = 0.$$

Thus $\kappa x + y \in \mathcal{N}$.

$\mathcal{W}/\mathcal{N}$ is a vector space over $\mathbb{K}$ This is a result from elementary linear algebra theory, but let's do it for revision. It is clear that $\mathcal{N} \in \mathcal{W}/\mathcal{N}$, so $\mathcal{W}/\mathcal{N} \neq \emptyset$. Notice that for any $[x], [y] \in \mathcal{W}/\mathcal{N}$ and $\kappa \in \mathbb{K}$, we define the operations

$$[\kappa x + y] = \kappa[x] + [y].$$

By the commutativity of addition,

$$[x + y] = x + y + \mathcal{N} = y + x + \mathcal{N} = [y + x].$$

The additive identity is $[0] = 0 + \mathcal{N}$, multiplicative identity is

$[1] = 1 + \mathcal{N}$, and additive inverse of $[x]$ is $[-x]$.

We note that $\mathcal{W}/\mathcal{N}$ is normally referred to as the **quotient space** of $\mathcal{W}$ by $\mathcal{N}$.

$\|\cdot\|$ is well-defined Let $[x_1] = [x_2] \in \mathcal{W}/\mathcal{N}$. Then $[x_1 - x_2] = [0]$ and so $x_1 - x_2 \in \mathcal{N}$. Then

$$\nu(x_1 - x_2) = 0,$$

which then since

$$0 \leq |\nu(x_1) - \nu(x_2)| \leq \nu(x_1 - x_2) = 0,$$

we have that $\nu(x_1) = \nu(x_2)$. Hence

$$\|[x_1]\| = \|[x_2]\|,$$

and so $\nu(\cdot)$ is well-defined.

$\|\cdot\|$ is a norm Let $[x], [y] \in \mathcal{W}/\mathcal{N}$ and $\kappa \in \mathbb{K}$. Then

- $\|[x]\| = \nu(x) \geq 0$;
- $\|\kappa[x]\| = \|[\kappa x]\| = \nu(\kappa x) = |\kappa| \, \nu(x) = |\kappa| \, \|[x]\|$;
- $\|[x] + [y]\| = \|[x + y]\| = \nu(x + y) \leq \nu(x) + \nu(y) = \|[x]\| + \|[y]\|$; and
- $\|[x]\| = 0 \implies \nu(x) = 0 \implies x \in \mathcal{N} \iff [x] = [0] \in \mathcal{W}/\mathcal{N}$.

Thus $\|\cdot\|$ is indeed a norm.

Hence, $\mathcal{W}/\mathcal{N}$ is a normed linear space. □

Example 10.2.2

In our last example, we determined that $\nu_1(\cdot)$ is a seminorm on $\mathcal{L}_1(E, \mathbb{K})$. Suppose

$$g \in \mathcal{N}_1(E, \mathbb{K}) := \{f \in \mathcal{L}_1(E, \mathbb{K}) : \nu_1(f) = 0\}.$$

Then $\int_E |g| = 0$. Since $mE > 0$, this happens if and only if $g = 0$ a.e. on E.

Since $g = 0$ a.e. on E iff $\int_E |g| = 0$, we can also define

$$\mathcal{N}_1(E, \mathbb{K}) = \{g \in \mathcal{L}_1(E, \mathbb{K}) : g = 0 \text{ a.e. on } E\}.$$

Setting

$$L_1(E, \mathbb{K}) = \mathcal{L}_1(E, \mathbb{K}) / \mathcal{N}_1(E, \mathbb{K}),$$

we have that $[f] = [g]$ iff $f - g \in \mathcal{N}_1(E, \mathbb{K})$, i.e. $f = g$ a.e. on E.

Definition 32 (L_1-space)

Let $E \in \mathfrak{M}(\mathbb{K})$ with $mE > 0$. We define the L_1-space as

$$L_1(E, \mathbb{K}) := \mathcal{L}_1(E, \mathbb{K}) / \mathcal{N}_1(E, \mathbb{K}),$$

with the norm

$$\begin{aligned} \|\cdot\| : L_1(E, \mathbb{K}) &\to \mathbb{R} \\ \|[f]\| &:= \int_E f. \end{aligned}$$

Definition 33 ($\mathcal{L}_p(E, \mathbb{K})$)

Let $E \in \mathfrak{M}(\mathbb{K})$ with $mE > 0$. If $1 < p < \infty$ in $\mathbb{R}$, we define

$$\begin{aligned} \mathcal{L}_p(E, \mathbb{K}) &:= \{f \in \mathcal{L}(E, \mathbb{K}) : \int_E |f|^p < \infty\} \\ &= \{f \in \mathcal{L}(E, \mathbb{K}) : |f|^p \in \mathcal{L}_1(E, \mathbb{K})\}. \end{aligned}$$

We need to show that $\mathcal{L}_p(E, \mathbb{K})$ is a vector space for all $1 < p < \infty$, and that

$$\nu_p(f) := \left(\int_E |f|^p \right)^{\frac{1}{p}}$$

defines a semi-norm on $\mathcal{L}_p(E, \mathbb{K})$. If we can establish these results, we can then appeal to Proposition 44 and take the quotient space wrt to a similar kernel.

However, the proof of the triangle inequality of ν_p is a non-trivial exercise.

Definition 34 (Lebesgue Conjugate)

Let $1 \leq p \leq \infty$. We associate to p the number $1 \leq q \leq \infty$ as follows:

- *if $p = 1$, then $q = \infty$;*
- *if $p = \infty$, then $q = 1$; and finally*
- $1 < p < \infty \implies$
$$q = \left(1 - \frac{1}{p}\right)^{-1}.$$

We say that q is the ***Lebesgue conjugate*** *of p. With the convention that $\frac{1}{\infty} := 0$, and we see that in all cases,*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Note 10.2.1

When $1 < p < \infty$, we see that the above equation is equivalent to each of the equations:

- *$p(q-1) = q$ and*
- *$(p-1)q = p$.*

Lemma 45 (Young's Inequality)

If $1 < p < \infty$ and q is the Lebesgue conjugate of p, then for $0 < a, b \in \mathbb{R}$,

1. *$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$; and*
2. *equality in the above holds iff $a^p = b^q$.*

There's another proof that I prefer over this construction here that feels like we just lucked out. See PMATH351.

Proof

Let $g : (0, \infty) \to \mathbb{R}$ be such that

$$x \mapsto \frac{1}{p} x p^{+} \frac{1}{q} - x.$$

Notice that g is differentiable on $(0, \infty)$, and we have

$$g'(x) = x^{p-1} - 1.$$

Furthermore,

- $g'(x) < 0$ for $x \in (0, 1)$;
- $g'(1) = 0$; and
- $g'(x) > 0$ for $x \in (1, \infty)$.

Also, note that $g(1) = \frac{1}{p} + \frac{1}{q} - 1 = 0$. Thus by the above observation, we know that g attains its minimum at 1. Let $x_0 = \frac{a}{b^{q-1}} > 0$. Then we have

$$\begin{aligned} 0 \leq g(x_0) &= \frac{1}{p} \left(\frac{a^p}{b^{(q-1)p}} \right) + \frac{1}{q} - \frac{a}{b^{q-1}} \\ &= \frac{1}{p} \frac{a^p}{b^q} + \frac{1}{q} - \frac{a}{b^{q-1}}. \end{aligned}$$

Thus

$$\frac{a}{b^{q-1}} \leq \frac{1}{p} \frac{a^p}{b^q} + \frac{1}{q}.$$

Multiplying both sides by b^q, we get

$$ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q.$$

Furthermore, we notice that

$$g(x_0) = 0 \iff x_0 = 1 \iff a = b^{q-1} \iff a^p = b^{p(q-1)} = b^q. \quad \square$$

11 Lecture 11 Jun 11th 2019

11.1 L_p Spaces Continued

Theorem 46 (Hölder's Inequality)

Let $E \in \mathfrak{M}(\mathbb{R})$, $1 < p < \infty$ in $\mathbb{R}$, and let q be the Lebesgue conjugate of p. Then

1. *If $f \in \mathcal{L}_p(E, \mathbb{K})$ and $g \in \mathcal{L}_q(E, \mathbb{K})$, then $fg \in \mathcal{L}_1(E, \mathbb{K})$ and*

$$\nu_1(fg) \leq \nu_p(f)\nu_q(g),$$

   where

$$\nu_p(f) = \left(\int_E |f|^p \right)^{\frac{1}{p}} \text{ and } \nu_q(g) = \left(\int_E |g|^q \right)^{\frac{1}{q}}$$

2. *Suppose that $H := \{x \in E : f(x) \neq 0\}$ has positive measure. If*

$$f^* := \nu_p(f)^{1-p}\overline{\Theta}\,|f|^{p-1},$$

   which is called the ***Lebesgue conjugate function****, then $f^* \in \mathcal{L}_q(E, \mathbb{K})$, $\nu_q(f) = 1$, and*

$$\nu_1(ff^*) = \int_E ff^* = \nu_p(f).$$

Proof

1. If $f = 0$ or $g = 0$ a.e. on E, then the inequality is trivially true. So wma $f \neq 0 \neq g$ a.e. on E. Now, for any $\alpha, \beta \in \mathbb{K}$, $\alpha f \in \mathcal{L}_p(E, \mathbb{K})$

and $\beta g \in \mathcal{L}_q(E, \mathbb{K})$ since

$$\int_E \alpha f = \alpha \int_E f < \infty$$

and

$$\int_E \beta g = \beta \int_E g < \infty.$$

Supposing that we can find $\alpha_0 \neq 0 \neq \beta_0$ such that

$$\int_E |(\alpha_0 f)(\beta_0 g)| \leq \nu_p(\alpha_0 f)\nu_q(\beta_0 g),$$

we see that we can factor out α_0 and β_0 so that

$$|\alpha_0 \beta_0| \int_E |fg| \leq |\alpha_0 \beta_0| \, \nu_p(f)\nu_q(g),$$

which then

$$\int_E |fg| \leq \nu_p(f)\nu_q(g).$$

Thus, choosing $\alpha_0 = \nu_p(f)^{-1}$ and $\beta_0 = \nu_q(g)^{-1}$, wma wlog $\nu_p(f) = 1 = \nu_q(g)$.

Now, by Lemma 45, we obtain

$$|fg| \leq \frac{|f|^p}{p} + \frac{|g|^q}{q}.$$

Thus

$$\begin{aligned}
\nu_1(fg) = \int_E |fg| &\leq \frac{1}{p} \int_E |f|^p + \frac{1}{q} \int_E |g|^q \\
&= \frac{1}{p}\nu_p(f)^p + \frac{1}{q}\nu_q(g)^q \\
&= \frac{1}{p} \cdot 1 + \frac{q}{1} \cdot 1 \\
&= 1 = \nu_p(f)\nu_q(g).
\end{aligned}$$

2. First, note that f^* is measurable, since $f, |f|$ and Θ are all measurable (cf. Proposition 23 and Proposition 22). Since $(p-1)q = p$, we have [1]

$$\begin{aligned}
\nu_q(f^*)^q = \int_E |f^*|^q &= \int_E \left(\nu_p(f)^{1-p} \, |f|^{p-1} \right)^q \\
&= \nu_p(f)^{-(p-1)q} \int_E |f|^{(p-1)q}
\end{aligned}$$

[1] **How did Θ disappear?**

$$= \nu_p(f)^{-p}\nu_p(f)^p = 1.$$

Finally,

$$\begin{aligned}\nu_1(ff^*) &= \int_E |ff^*| = \int_E \nu_p(f)^{1-p} |f|^{p-1} |f| \\ &= \nu_p(f)^{1-p} \int_E |f|^p \\ &= \nu_p(f)^{1-p}\nu_p(f)^p \\ &= \nu_p(f).\end{aligned}$$

□

Theorem 47 (Minkowski's Inequality)

Let $E \in \mathfrak{M}(\mathbb{R})$, $1 < p < \infty$. If $f, g \in \mathcal{L}_p(E, \mathbb{K})$, then $f + g \in \mathcal{L}_p(E, \mathbb{K})$ and

$$\nu_p(f+g) \leq \nu_p(f) + \nu_p(g).$$

Proof

$f + g$ is measurable by Proposition 22. Notice that for $0 \leq a, b$, we have

$$(a+b)^p \leq (2\max\{a,b\})^p \leq 2^p(a^p + b^p).$$

Thus

$$|f+g|^p \leq (|f| + |g|)^p \leq 2^p \left(|f|^p + |g|^p\right).$$

It follows that

$$\nu_p(f+g) = \int_E |f+g|^p \leq 2^p \left(\nu_p(f)^p + \nu_p(g)^p\right) < \infty.$$

Thus $f + g \in \mathcal{L}_p(E, \mathbb{K})$.

Now let $h = f + g$, and h^* the Lebesgue conjugate function of h. Then $h^* \in \mathcal{L}_q(E, \mathbb{K})$. By the last theorem, $\nu_q(h) = 1$ and $\nu_1(hh^*) = \nu_p(h)$. With this, and Hölder's Inequality, we have

$$\begin{aligned}\nu_p(f+g) &= \nu_p(h) = \nu_1(hh^*) \\ &= \nu_1((f+g)h^*)\end{aligned}$$

$$
\begin{aligned}
&\leq \nu_1(fh^*) + \nu_1(gh^*) \\
&\overset{(*)}{\leq} \nu_p(f)\nu_q(h^*) + \nu_p(g)\nu_q(h^*) \\
&= \nu_p(f) + \nu_p(g),
\end{aligned}
$$

where $(*)$ is where we use Hölder's Inequality. □

We are finally ready to show that $\mathcal{L}_p(E, \mathbb{K})$ is a vector space and ν_p is a semi-norm as claimed.

Corollary 48 (ν_p is a Semi-Norm)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $1 < p < \infty$. Then $\mathcal{L}_p(E, \mathbb{K})$ is a vector space over $\mathbb{K}$ and ν_p defines a semi-norm on $\mathcal{L}_p(E, \mathbb{K})$.

Proof

$\mathcal{L}_p(E, \mathbb{K})$ is a vector space Since $\mathbb{K}$ is a vector space, we need only check that $\mathcal{L}_p(E, \mathbb{K})$ is nonempty, and closed under addition and scalar multiplication.

$\mathcal{L}_p(E, \mathbb{K}) \neq \emptyset$ It is clear that the constant function, $f(x) = 0$ for all $x \in E$, is in $\mathcal{L}_p(E, \mathbb{K})$ since

$$\int_E f = \int_E 0 = 0 < \infty.$$

$\mathcal{L}_p(E, \mathbb{K})$ is closed under addition and scalar multiplication Let $f, g \in \mathcal{L}_p(E, \mathbb{K})$ and $\kappa \in \mathbb{K}$. Then by Minkowski's Inequality,

$$\nu_p(\kappa f + g) \leq \nu_p(\kappa f) + \nu_p(g) = |\kappa| \, \nu_p(f) + \nu_p(g) < \infty.$$

ν_p is a semi-norm We showed for the first two conditions and MinkMinkowski's Inequality covers the Triangle Inequality. □

Definition 35 (L_p-Space and L_p-Norm)

*Let $E \in \mathfrak{M}(\mathbb{R})$ and $1 < p < \infty$. We define the **L_p-space***

$$L_p(E, \mathbb{K}) := \mathcal{L}_p(E, \mathbb{K}) / \mathcal{N}_p(E, \mathbb{K}),$$

where [2]

$$\mathcal{N}_p(E, \mathbb{K}) = \{f \in \mathcal{L}_p(E, \mathbb{K}) : \nu_p(f) = 0\}.$$

[2] Note that $\mathcal{N}_p(E, \mathbb{K})$ is where functions are 0 a.e.

*The **L_p-norm** on $L_p(E, \mathbb{K})$ is the norm defined by*

$$\begin{aligned} \|\cdot\|_p : L_p(E, \mathbb{K}) &\to \mathbb{R} \\ [f] &\mapsto \nu_p(f). \end{aligned}$$

For the sake of completeness, we shall restate Hölder's and Minkowski's Inequalities for $L_p(E, \mathbb{K})$.

Theorem 49 (Hölder's Inequality)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $1 < p < \infty$. Let q denote the Lebesgue conjugate of p.

1. *If $[f] \in L_p(E, \mathbb{K})$ and $[g] \in L_q(E, \mathbb{K})$, then $[f][g] := [fg] \in L_1(E, \mathbb{K})$ is well-defined and*

$$\|[fg]\|_1 \leq \|[f]\|_p \|[g]\|_q .$$

2. *If $0 \neq [f] \in L_p(E, \mathbb{K})$ and f^* is the conjugate function of f, then $[f^*] \in L_q(E, \mathbb{K})$, $\|[f^*]\|_q = 1$, and*

$$\|[f][f^*]\| = \|[f]\|_p .$$

Proof

The only part that does not follow immediately from Theorem 46 is the well-definedness of $[f][g] = [fg]$. □

Theorem 50 (Minkowski's Inequality)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $1 < p < \infty$. If $[f], [g] \in L_p(E, \mathbb{K})$, then $[f+g] \in L_p(E, \mathbb{K})$ and

$$\|[f+g]\|_p = \|[f]+[g]\|_p \leq \|[f]\|_p + \|[g]\|_p .$$

We can now show that $L_p(E, \mathbb{K})$ is a Banach space for all $1 \leq p < \infty$, whose proof shall be left for next lecture.

12 Lecture 12 Jun 18th 2019

12.1 L_p Spaces (Continued 2)

Theorem 51 (($L_p(E, \mathbb{K}), \|\cdot\|_p$) is Banach Space)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $1 \leq p < \infty$. Then $L_p(E < \mathbb{K})$ is complete and hence Banach.

Strategy

By Proposition 44, $(L_p(E, \mathbb{K}), \|\cdot\|_p)$ is a normed linear space. It thus suffices for us to show that it is complete.

This is a preferred approach by the professor, that he has defaulted to proving completeness from the equivalent result of having every absolutely summable series being summable in the space. We prove this equivalence in A4.

So given an absolutely summable sequence $\{[f_n]\}_{n=1}^{\infty}$, since we want

$$\sum_{n=1}^{\infty} [f_n] < \infty \text{ a.e. on } E,$$

in particular this should be reflected by any of its representatives, i.e. if we take, wlog, f_n as the representative of $[f_n]$, then we want

$$h = \sum_{n=1}^{\infty} f_n < \infty \text{ a.e. on } E.$$

To show that the sum is finite a.e. on E, we will first make use of the fact *that this would be equivalent to*

$$|h| = \left| \sum_{n=1}^{\infty} f_n \right| < \infty \text{ a.e. on } E.$$

To that end, the partial sums should always be finite. By the triangle inequality, we see that

$$\left| \sum_{n=1}^{N} f_n \right| \leq \sum_{n=1}^{N} |f_n| .$$

This is where our 'clean' proof begins.

Proof

Suppose $\{[f_n]\}_{n=1}^{\infty}$ is a sequence of equivalence classes in $L_p(E, \mathbb{K})$ that is absolutely summable. We note that the following value will be useful, and so we give it a variable.

$$\gamma \coloneqq \sum_{i=1}^{\infty} \|[f_n]\|_p .$$

Showing that $\sum_{n=1}^{\infty} f_n(x)$ converges a.e. on E For each $N \geq 1$, let $g_N = \sum_{n=1}^{N} |f_n|$. Note that since $f_n \in \mathcal{L}_p(E, \mathbb{K})$, by Corollary 48, we have that $g_N \in \mathcal{L}_p(E, [0, \infty])$. Furthermore, since g_N is a sum of absolute values, we have that

$$0 \leq g_1 \leq g_2 \leq g_3 \leq \dots .$$

Let $g_\infty \coloneqq \lim_{N\to\infty} g_N = \sup_{N\geq 1} g_N$. By Proposition 27, $g_\infty \in \mathcal{L}(E, [0, \infty])$. [1] Note that $g_\infty^p = \sup_{N\geq 1} g_N^p$. By the Monotone Convergence Theorem, we observe that

$$\begin{aligned}
\int_E g_\infty^p &= \lim_{N\to\infty} \int_E g_N^p \\
&= \lim_{N\to\infty} \int_E \left(\sum_{n=1}^{N} |f_n| \right)^p \\
&= \lim_{N\to\infty} \int_E \\
&= \lim_{N\to\infty} \nu_p \left(\sum_{n=1}^{N} |f_n| \right)^p
\end{aligned}$$

[1] Now, we want to show that even $g_\infty < \infty$ a.e. on E. Following this is a non-trivial step forward.

$$\leq \lim_{N\to\infty}\left(\sum_{n=1}^{N}\nu_p(|f_n|)\right)^p$$
$$\leq \left(\sum_{n=1}^{\infty}\|[f_n]\|_p\right)^p = \gamma^p < \infty$$

by assumption. Thus $g_\infty \in \mathcal{L}_p(E,\mathbb{K})$, which means that $g_\infty < \infty$ a.e. on E. From here, we observe that

$$\left|\sum_{n=1}^{\infty} f_n\right| \leq \sum_{n=1}^{\infty}|f_n| \leq g_\infty \leq \gamma < \infty.$$

Then since $\mathbb{K}$ is complete, $\sum_{n=1}^{\infty} f_n(x)$ converges to some value in $\mathbb{K}$ for every $x \in E$.

Constructing $h = \sum_{n=1}^{\infty} f_n$ a.e. on E In particular, we want the above sum to converge to some function $h = \sum_{n=1}^{\infty} f_n$ a.e. on E. We want to explicitly isolate the points where the sum goes bad. Letting

$$B := \{x \in E : g_\infty(x) = \infty\} \subseteq E,$$

we have that $mB = 0$. Consider $H = E \setminus B \in \mathfrak{M}(E)$. [2] Here, let $g = \chi_H \cdot g_\infty$. Note that since $H \in \mathfrak{M}(E)$, χ_H is measurable, and so by Proposition 22, $g \in \mathcal{L}(E,[0,\infty))$, and $g = g_\infty$ a.e. on E. Furthermore,

$$\int_E g^p = \int_E g_\infty^p \leq \gamma^p,$$

and so $g \in \mathcal{L}_p(E,[0,\infty)) \subseteq L_p(E,\mathbb{K})$, i.e. $[g] \in L_p(E,\mathbb{K})$ and $\|[g]\|_p \leq \gamma$.

[2] We will build h on this nicer set.

For each $N \geq 1$, let $h_N := \chi_H \cdot \left(\sum_{n=1}^{N} f_n\right)$. By the same reasoning as for g, we have that $h_N \in \mathcal{L}_p(E,\mathbb{K}) \subseteq \mathcal{L}(E,\mathbb{K})$. Moreover, it is clear from construction that $[h_N] = \sum_{n=1}^{N}[f_n]$, since $h_N = \sum_{n=1}^{N} f_n$ a.e. on E, in particular, they agree on H. It is also important to note that for $x \in H$,

$$|h_N(x)| \leq \sum_{n=1}^{N}|f_n(x)| \leq g(x),$$

and for $x \in B$, $|h_N(x)| = 0 = g(x)$. Thus $|h_N| < g$, and so $|h_N|^p \leq g^p$. So for each $N \geq 1$, we have

$$\int_E |h_N|^p \leq \int_E g^p \leq \gamma^p.$$

Since the partials are all well-defined, we can define

$$h(x) := \lim_{N\to\infty} h_N(x) \in \mathbb{K} \text{ for } x \in E.$$

Again, by Proposition 27, $h \in \mathcal{L}(E, \mathbb{K})$. Furthermore, since each $|h_N| \leq g$, we have that $|h| \leq g$ and $|h|^p \leq g^p$, which then

$$\int_E |h|^p \leq \int_E g^p \leq \gamma^p < \infty.$$

It follows that $h \in \mathcal{L}_p(E, \mathbb{K})$ and $[h] \in L_p(E, \mathbb{K})$.

$[h] = \lim_{N\to\infty}[h_N]$ It remains for us to show that this equation is true. In other words, we want to show that

$$\lim_{N\to\infty} \|[h] - [h_N]\|_p = \lim_{N\to\infty} \left\| [h] - \sum_{n=1}^{N} [f_n] \right\|_p = 0.$$

Note that $|h_M - h_N|^p \leq (|h_M| + |h_N|)^p \leq (g+g)^p$ for any M, N, and $\int_E (2\,|g|)^p < \infty$. Then, satisfying the condition for the Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}
\|[h] - [h_N]\|_p &= \nu_p(h - h_N) \\
&= \left(\int_E |h - h_N|^p \right)^{\frac{1}{p}} \\
&= \left(\int_E \lim_{M\to\infty} |h_M - h_N|^p \right)^{\frac{1}{p}} \\
&= \left(\lim_{M\to\infty} \int_E |h_M - h_N|^p \right)^{\frac{1}{p}} \\
&= \lim_{M\to\infty} \left(\int_E |h_M - h_N|^p \right)^{\frac{1}{p}} \\
&= \lim_{M\to\infty} \|[h_M] - [h_N]\|_p \\
&= \lim_{M\to\infty} \left\| \sum_{n=N+1}^{M} [f_n] \right\|_p \\
&\leq \lim_{M\to\infty} \sum_{n=N+1}^{M} \|[f_n]\|_p \\
&= \sum_{n=N+1}^{\infty} \|[f_n]\|_p
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \|[f_n]\|_p = \gamma^p < \infty$ by assumption, we have that

$$\lim_{N\to\infty} \|[h] - [h_N]\|_p = \lim_{N\to\infty} \sum_{n=N+1}^{\infty} \|[f_n]\|_p = 0.$$

This completes the proof. □

Notice that in Theorem 51 we talked about $1 \leq p < \infty$ but not $p = \infty$ itself. We shall explore this in the following subsection.

12.1.1 *Completeness of $L_\infty(E, \mathbb{K})$*

We need to first clarify what the norm in $L_\infty(E, \mathbb{K})$ is. It would be sensible to immediately let the norm be the supremum of the function, but we want to exclude the places where f hit its 'suprema' only up to a set of measure zero.

Definition 36 (Essential Supremum)

Let $E \in \mathfrak{M}(\mathbb{R})$ and $f \in \mathcal{L}(E, \mathbb{K})$. We define the ***essential supremum*** *of f on E as*

$$\nu_\infty(f) := \inf\{\gamma > 0 : m(\{x \in E : |f(x)| > \gamma\}) = 0\}.$$

Note 12.1.1

1. *Let us try to describe the essential supremum in words: we pick out the smallest γ (specifically, we pick the inf) such that the places on E where $f > \gamma$ is measure zero. Graphically, we set lower and lower γ until we finally hit some value where $f > \gamma$ but the places where this happens is no longer of measure zero.*
2. *Simply by definition, we have that $\nu_\infty(f) \geq 0$ for any $f \in \mathcal{L}(E, \mathbb{K})$.*

Definition 37 ($\mathcal{L}_\infty(E, \mathbb{K})$)

With the essential supremum, we can define

$$\mathcal{L}_\infty(E, \mathbb{K}) = \{f \in \mathcal{L}(E, \mathbb{K}), \nu_\infty(f) < \infty\}.$$

Example 12.1.1

1. Let $E = \mathbb{R}$ and $f = \chi_{\mathbb{Q}}$. Observe that for any $\gamma > 0$, since

$$\{x \in \mathbb{R} : |\chi_{\mathbb{Q}}| > \gamma\} \subseteq \mathbb{Q},$$

we have

$$0 \leq m\{x \in \mathbb{R} : |\chi_{\mathbb{Q}}| > \gamma\} \leq m\mathbb{Q} = 0.$$

Thus $\nu_\infty(\chi_{\mathbb{Q}}) = 0$.

Note that there was nothing special about the choice of $\mathbb{Q}$ except that it is a set of measure zero.

2. Suppose $a < b \in \mathbb{R}$ and $f \in \mathcal{C}([a, b], \mathbb{K})$.

Claim: $f \in \mathcal{L}_\infty([a, b], \mathbb{K})$ and $\nu_\infty(f) = \|f\|_{\sup} := \sup_{x \in [a,b]|f(x)|}$
We know that every continuous function on a measurable set is measurable [3], so $f \in \mathcal{L}([a, b], \mathbb{K})$.

[3] cf. Proposition 19

Note that for $\gamma > \|f\|_{\sup}$, we have that

$$m\left(\{x \in [a, b] : |f(x)| > \gamma\}\right) = m(\emptyset) = 0.$$

So $\nu_\infty(f) \leq \gamma$. Since this holds for all γ, it follows that $\nu_\infty(f) \leq |f|_{\sup}$.

On the other hand, for $\gamma \leq \|f\|_{\sup} = |f(x_0)|$ for some $x_0 \in [a, b]$. By continuity of f on $[a, b]$, and in particular on x_0, $\exists \delta > 0$ such that $\forall x \in (x_0 - \delta, x_0 + \delta) \cap [a, b]$ implies that $|f(x)| > \gamma$. Notice that

$$m\left((x_0 - \delta, x_0 + \delta) \cap [a, b]\right) > 0,$$

which means that

$$\nu_\infty(f) \geq \gamma.$$

This holds for all γ, and so

$$\nu_\infty(f) \geq \|[f]\|_{\sup}.$$

Thus

$$\nu_\infty(f) = \|[f]\|_{\sup},$$

which also gives us that

$$f \in \mathcal{L}_\infty([a,b], \mathbb{K}).$$

Proposition 52 ($\mathcal{L}_\infty(E, \mathbb{K})$ is a vector space and $\nu_\infty(\cdot)$ a semi-norm)

Let $E \in \mathfrak{M}(\mathbb{R})$. Then $\mathcal{L}_\infty(E, \mathbb{K})$ is a vector space over $\mathbb{K}$ and $\nu_\infty(\cdot)$ is a semi-norm on $\mathcal{L}_\infty(E, \mathbb{K})$.

Proof

Since $\mathcal{L}_\infty(E, \mathbb{K}) \subseteq \mathcal{L}(E, \mathbb{K})$, and that the latter is a vector space, it suffices to perform the subspace test on $\mathcal{L}_\infty(E, \mathbb{K})$ to show that $\mathcal{L}_\infty(E, \mathbb{K})$ is a vector space.

First, note that if $\zeta = 0$ is the zero function, then $\nu_\infty(\zeta) = 0 < \infty$, and so $\zeta \in \mathcal{L}_\infty(E, \mathbb{K})$, i.e. $\mathcal{L}_\infty(E, \mathbb{K}) \neq \emptyset$. Further, as noted before, $\nu_\infty(f) \geq 0$ for any $f \in \mathcal{L}_\infty(E, \mathbb{K})$.

Next, suppose that $f \in \mathcal{L}_\infty(E, \mathbb{K})$ and $0 \neq \kappa \in \mathbb{K}$. It is clear that $\kappa f \in \mathcal{L}(E, \mathbb{K})$, and we quickly notice that

$$\begin{aligned}
\nu_\infty(\kappa f) &= \inf\{\gamma > 0 : m\{x \in E : |\kappa f(x)| > \gamma\} = 0\} \\
&= \inf\{|\kappa|\,\delta : m\{x \in E : |\kappa|\,|f(x)| > |\kappa|\,\delta\} = 0\} \\
&= |\kappa| \inf\{\delta > 0 : m\{x \in E : |f(x)| > \delta\} = 0\} \\
&= |\kappa|\,\nu_\infty(f) < \infty.
\end{aligned}$$

So $\kappa f \in \mathcal{L}_\infty(E, \mathbb{K})$ for all $0 \neq \kappa \in \mathbb{K}$. As noted before, if $\kappa = 0$, then $\kappa f = 0 \in \mathcal{L}_\infty(E, \mathbb{K})$.

Now suppose $f, g \in \mathcal{L}_\infty(E, \mathbb{K})$. WTS

$$\nu_\infty(f+g) \leq \nu_\infty(f) + \nu_\infty(g).$$

Let $\alpha > \nu_\infty(f)$ and $\beta > \nu_\infty(g)$. Let

$$E_f = \{x \in E : |f(x)| > \alpha\} \text{ and } E_g = \{x \in E : |g(x)| > \beta\}.$$

Then $mE_f = 0 = mE_g$. Let $H = E \setminus (E_f \cup E_g)$. For $x \in H$, we have

$$|(f+g)(x)| \leq |f(x)| + |g(x)| \leq \alpha + \beta,$$

so

$$\{x \in E : |(f+g)(x)| > \alpha + \beta\} \subseteq E_f \cup E_g.$$

Thus

$$m\{x \in E : |(f+g)(x)| > \alpha + \beta\} \leq mE_f + mE_g = 0,$$

and so $\nu_\infty(f+g) \leq \alpha + \beta$. Since α and β were arbitrary, it follows that

$$\nu_\infty(f+g) \leq \nu_\infty(f) + \nu_\infty(g) < \infty.$$

Thus $\mathcal{L}_\infty(E, \mathbb{K})$ and $\nu_\infty(\cdot)$ is indeed a semi-norm. □

Definition 38 ($L_\infty(E, \mathbb{K})$)

Let

$$\mathcal{N}_\infty(E, \mathbb{K}) := \{f \in \mathcal{L}_\infty(E, \mathbb{K}) : \nu_\infty(f) = 0\}.$$

Then we define

$$L_\infty(E, \mathbb{K}) = \mathcal{L}_\infty(E, \mathbb{K}) / \mathcal{N}_\infty(E, \mathbb{K}),$$

and we denote by $[f]$ the coset of $f \in \mathcal{L}_\infty(E, \mathbb{K})$ in $\mathcal{L}_\infty(E, \mathbb{K})$.

Theorem 53 ($L_\infty(E, \mathbb{K})$ is a normed-linear space)

Let $E \in \mathfrak{M}(\mathbb{R})$. Then $L_\infty(E, \mathbb{K})$ is a normed-linear space, where for

$[f] \in L_\infty(E, \mathbb{K})$ we set

$$\|[f]\|_\infty := \nu_\infty(f).$$

Proof

See Proposition 44. □

Remark 12.1.1

Let $f \in \mathcal{L}_\infty(E, \mathbb{K})$. Let us look at the places where the undesirable happens. For each $n \geq 1$, let

$$B_n := \left\{ x \in E : |f(x)| > \nu_\infty(f) + \frac{1}{n} \right\}.$$

Then by definition of $\nu_\infty(\cdot)$, we have that $mB_n = 0$ for each $n \geq 1$, and letting

$$B := \bigcup_{n=1}^{\infty} B_n = \{ x \in E : |f(x)| > \nu_\infty(f) \},$$

we have that

$$mB \leq \sum_{n=1}^{\infty} mB_n = \sum_{n=1}^{\infty} 0 = 0.$$

In other words, for any $f \in \mathcal{L}_\infty(E, \mathbb{K})$, the set

$$B = \{ x \in E : |f(x)| > \nu_\infty(f) \}$$

has measure zero. So for any $[f] \in L_\infty(E, \mathbb{K})$, we can always pick a representative $g \in [f]$ such that

$$|g(x)| \leq \|[f]\|_\infty$$

for all $x \in E$.

In particular, the function $g := \chi_{E \setminus B} \cdot f$ is measurable, and differs from f only on B, whence $[g] = [f]$, and we indeed have

$$|g(x)| \leq \nu_\infty(f) = \nu_\infty(g) = \|[g]\|_\infty$$

for all $x \in E$.

Moreover, we see that $\nu_\infty(f) = 0$ iff $f = 0$ a.e. on E, and so

$$\mathcal{N}_\infty(E, \mathbb{K}) = \{f \in \mathcal{L}_\infty(E, \mathbb{K}) : f = 0 \text{ a.e. on } E\}.$$

Theorem 54 (Completeness of $L_\infty(E, \mathbb{K})$)

Let $E \in \mathfrak{M}(\mathbb{R})$. Then $L_\infty(E, \mathbb{K})$ is a Banach space.

Proof

To be added □

Recall that if $E \in \mathfrak{M}(\mathbb{R})$ and $1 < p < \infty$, $f \in \mathcal{L}_p(E, \mathbb{K})$ and $g \in \mathcal{L}_q(E, \mathbb{K})$, where q is the Lebesgue conjugate of p, then Hölder's Inequality gives that $fg\, in \mathcal{L}_1(E, \mathbb{K})$ and

$$\nu_1(fg) \leq \nu_p(f)\nu_q(g).$$

Let's look at $p = 1$.

Theorem 55 (Hölder's Inequality for $\mathcal{L}_1(E, \mathbb{K})$)

Let $E \in \mathfrak{M}(\mathbb{R})$ with $mE > 0$.

1. *If $f \in \mathcal{L}_1(E, \mathbb{K})$ and $g \in \mathcal{L}_\infty(E, \mathbb{K})$, then $fg \in \mathcal{L}_1(E, \mathbb{K})$ and*

$$\nu_1(fg) \leq \nu_1(f)\nu_\infty(g).$$

2. *For $f \in \mathcal{L}_1(E, \mathbb{K})$, there exists a function $f^* \in \mathcal{L}_\infty(E, \mathbb{K})$ such that $\nu_\infty(f^*) = 1$ and*

$$\nu_1(ff^*) = \int_E f \cdot f^* = \nu_1(f).$$

Proof

1. By Remark 12.1.1, for $[g] \in L_\infty(E, \mathbb{K})$, we can find, wlog, $g_0 \in [g]$ so that $g_0 = g$ a.e. on E, and for all $x \in E$, we have $|g_0(x)| \leq \nu_\infty(g) = \nu_\infty(g_0)$. In particular, we have that for any $f \in \mathcal{L}_1(E, \mathbb{K})$, $|fg| = |fg_0|$ a.e. on E, and we find that

$$\int_E |fg| = \int_E |fg_0| .$$

   Thus wlog wma $|g(x)| \leq \nu_\infty(g)$ for all $x \in E$. Then

$$\nu_1(fg) = \int_E |fg| \leq \int_E |f| \, \nu_\infty(g) = \nu_\infty(g) \int_E |f| = \nu_1(f)\nu_\infty(g).$$

2. Set $\Theta : E \to \mathbb{T}$ such that

$$f = \Theta \cdot |f| ,$$

   where

$$\Theta(x) = \begin{cases} \frac{f(x)}{|f(x)|} & \text{when } f(x) \neq 0 \\ 1 & \text{when } f(x) = 0. \end{cases}$$

   Then with $f^* := \overline{\Theta}$, we have $\nu_\infty(f^*) = 1$, $|f| = ff^*$, and so

$$\nu_1(ff^*) = \int_E |ff^*| = \int_E |f| = \nu_1(f). \qquad \square$$

Corollary 56 (Hölder's Inequality for $L_1(E, \mathbb{K})$)

Let $E \in \mathfrak{M}(\mathbb{R})$. If $[f] \in L_1(E, \mathbb{K})$ and $[g] \in \mathcal{L}_\infty(E, \mathbb{K})$, then $[f][g] := [fg] \in \mathcal{L}1(E, \mathbb{K})$ is well-defined and

$$\| [fg] \|_1 \leq \| [f] \|_1 \, \| [g] \|_\infty .$$

Corollary 57 (Hölder's Inequality for Continuous Functions)

Suppose that $a < b \in \mathbb{R}$. Consider $h \in \mathcal{C}([a, b], \mathbb{K})$ and $f \in \mathcal{L}_1([a, b], \mathbb{K})$. Then $h \cdot f \in \mathcal{L}_1([a, b], \mathbb{K})$ and

$$\nu_1(h \cdot f) \leq \nu_1(f)\nu_\infty(h) = \nu_1(f) \, \|h\|_{\sup} .$$

Proof

Continuous functions are measurable, so h is measurable, and $\mathcal{L}_\infty([a,b],\mathbb{K})$ with $\|h\|_{\sup} = \nu_\infty(h)$. Then it is simply Theorem 55. □

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13.1 L_p Spaces (Continued 3)

Remark 13.1.1 (Containment of L_p Spaces)

Let $E \in \mathfrak{M}(\mathbb{R})$ with $mE < \infty$. Suppose that $1 \leq p < \infty$, and that $[f] \in L_\infty(E, \mathbb{K})$, which then wlog $f \in \mathcal{L}_\infty(E, \mathbb{K})$. As commented before, $|f(x)| \leq \|[f]\|_\infty$ a.e. on E. Then

$$\|[f]\|_p = \int_E |f|^p \leq \int_E \|[f]\|_\infty^p = \|[f]\|_\infty^p \, mE < \infty,$$

which means $[f] \in L_p(E, \mathbb{K})$, with

$$\|[f]\|_p \leq \|[f]\|_\infty (mE)^{\frac{1}{p}}.$$

Thus $L_\infty(E, \mathbb{K}) \subseteq L_p(E, \mathbb{K})$, $1 \leq p < \infty$ when $mE < \infty$.

Next, consider $1 \leq p < r < \infty$. Suppose $[g] \in L_r(E, \mathbb{K})$. Again, wlog $g \in \mathcal{L}_r(E, \mathbb{K})$ and

$$\begin{aligned} \|[g]\|_p^p = \int_E |g|^p = \int_E (|g|^r)^{\frac{p}{r}} \leq \int_E \max\{1, |g|^r\} \\ \leq \int_E 1 + |g|^r = mE + \|[g]\|_r < \infty. \end{aligned}$$

So $[g] \in L_p(E, \mathbb{K})$. Thus we see that

$$L_\infty(E, \mathbb{K}) \subseteq L_r(E, \mathbb{K}) \subseteq L_p(E, \mathbb{K}) \subseteq L_1(E, \mathbb{K}).$$

Remark 13.1.2

Suppose $a < b \in \mathbb{R}$. Then from Example 12.1.1, we have that

$$[\mathcal{C}([a,b],\mathbb{K})] := \{[f] : f \in \mathcal{C}([a,b],\mathbb{K})\} \subseteq L_\infty([a,b]).$$

Recall that

$$\mathcal{R}_\infty([a,b],\mathbb{K}) = \{f : [a,b] \to \mathbb{K} : f \text{ is Riemann-integrable and bdd }\}.$$

By Corollary 41, $f \in \mathcal{L}([a,b],\mathbb{K})$ and so $[f] \in L_\infty([a,b],\mathbb{K})$ by virtue of f being bounded.

Our next goal is to establish that the space $[\mathcal{C}([a,b],\mathbb{K})]$ is dense in $L_p([a,b],\mathbb{K})$, for $1 \leq p < \infty$.

Definition 39 (Closed Span)

*We define the **closed span** of a subspace $\mathcal{B} \subseteq (\mathcal{H}, \|\cdot\|)$ as*

$$\overline{\operatorname{span}}\mathcal{B} := \{y \in \mathcal{H} : \forall \varepsilon > 0 \quad \exists x \in \operatorname{span}\mathcal{B} \quad \|x - y\| < \varepsilon\}$$

Imma use the name from the notes of Prof. Marcoux, 2019 for Lemma 58, since there's no good expressive name for it.

Lemma 58 (Lemma 6.31)

*Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space, and suppose that $\mathcal{Y}$ and $\mathcal{Z}$ are **linear manifolds** [1] in $\mathcal{X}$. Suppose $\mathcal{B} \subseteq \mathcal{Y}$ satisfies*

$$\overline{\operatorname{span}}\mathcal{B} = \mathcal{X}.$$

Then if $\mathcal{B} \subseteq \overline{\mathcal{Z}}$, then $\overline{\mathcal{Z}} = \mathcal{X}$.

[1] i.e. a vector subspace, but not necessarily closed.

Proof

Let $x \in \mathcal{X} = \overline{\operatorname{span}}\mathcal{B}$ and $\varepsilon > 0$. Then there exists $\{b_i\}_{i=1}^N \subseteq \mathcal{B}$ and $\{k_i\}_{i=1}^N \subseteq \mathbb{R}$ such that

$$\left\| x - \sum_{n=1}^N k_n b_n \right\| < \frac{\varepsilon}{2}.$$

Since $b_i \in \mathcal{B} \subseteq \overline{\mathcal{Z}}$, there exists $z_i \in \mathcal{Z}$ such that

$$\|z_i - b_i\| < \frac{\varepsilon}{2N(|k_i|+1)}.$$

Let $z := \sum_{n=1}^{N} k_n z_n \in \mathcal{Z}$, and this would give

$$\begin{aligned} \|x - z\| &\leq \left\| x - \sum_{n=1}^{N} k_n b_n \right\| + \left\| \sum_{n=1}^{N} k_n b_n - z \right\| \\ &< \frac{\varepsilon}{2} + \left\| \sum_{n=1}^{N} k_n (b_n - z_n) \right\| \\ &\leq \frac{\varepsilon}{2} + \sum_{n=1}^{N} |k_n| \, \|b_n - z_n\| \\ &\leq \frac{\varepsilon}{2} + \sum_{n=1}^{N} \frac{\varepsilon}{2N} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus $\mathcal{Z}$ is dense in $\mathcal{X}$. □

Notation

Let $E \in \mathfrak{M}(\mathbb{R})$ and $1 \leq p \leq \infty$. We set

$$\mathrm{SIMP}_p(E, \mathbb{K}) = \mathrm{SIMP}(E, \mathbb{K}) \cap \mathcal{L}_p(E, \mathbb{K}).$$

Exercise 13.1.1

Prove that if $mE < \infty$ or if $p = \infty$, then

$$\mathrm{SIMP}_p(E, \mathbb{K}) = \mathrm{SIMP}(E, \mathbb{K}).$$

Solution

Case $p = \infty$ By definition, a simple function f has finite range, and so $\nu_\infty(f) < \infty$. Thus $\mathrm{SIMP}(E, \mathbb{K}) \subseteq \mathcal{L}_\infty(E, \mathbb{K})$ and so our result holds.

Case $mE < \infty$ This is quite similar, especially since the range of f is finite, and so integration of a finite function over a finite domain is going to be finite. Thus, again $\mathrm{SIMP}(E, \mathbb{K}) \subseteq \mathcal{L}_p(E, \mathbb{K})$. ◎

Proposition 59 (Density of Equivalence Classes of $\mathrm{SIMP}_p(E, \mathbb{K})$ in $(L_p(E, \mathbb{K}), \|\cdot\|_p)$)

Let $E \in \mathfrak{M}(\mathbb{R})$ *be a Lebesgue measurable set and* $1 \leq p \leq \infty$. *Then*

$$[\mathrm{SIMP}_p(E, \mathbb{K})] := \{[\varphi] : \varphi \in \mathrm{SIMP}_p(E, \mathbb{K})\}$$

is dense in

$$(L_p(E, \mathbb{K}), \|\cdot\|_p).$$

Strategy

Recall Proposition 30. *This is the proposition that is key to showing that simple functions are dense, simply because we may get as close to any* $f \in \mathcal{L}(E, [0, \infty])$ *as we want.*

1. *Reduce to the problem to only real-valued functions.*
2. *Reduce the problem to only positive real-valued functions.*
3. *It then remains to reconstruct a simple function in* $\mathcal{L}_p(E, \mathbb{R})$ *that is as close to the original real-valued function as we would like.*

Proof

Case $\mathbb{K} = \mathbb{C}$ If we had proved the above for the case where $\mathbb{K} = \mathbb{R}$, then for $[g] \in L_p(E, \mathbb{K})$ and $\varepsilon > 0$, we may write

$$g = \Re g + i \Im g.$$

In particular, $g \in \mathcal{L}(E, \mathbb{C})$, so it is necessary that $\Re g$, $\Im g \in \mathcal{L}(E, \mathbb{R})$ by Proposition 21. Then, by assumption that this works for $\mathbb{R}$, we can pick $\varphi_1, \varphi_2 \in \mathrm{SIMP}(E, \mathbb{R})$ such that

$$\|[\Re g] - [\varphi_1]\|_p < \frac{\varepsilon}{2}$$
$$\|[\Im g] - [\varphi_2]\|_p < \frac{\varepsilon}{2}.$$

Then, let

$$\varphi = \varphi_1 + i\varphi_2 \in \mathrm{SIMP}(E, \mathbb{C}),$$

which then

$$\|[g] - [\varphi]\| \leq \|[\Re g] - [\varphi_1]\|_p + |i| \, \|[\Im g] - [\varphi_2]\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Then $[\mathrm{SIMP}(E, \mathbb{C})]$ is dense in $(L_p(E, \mathbb{C}), \|\cdot\|_p)$.

Case $\mathbb{K} = \mathbb{R}$ We shall further break this into 2 cases, of which we have seen in our last exercise.

Case 1: $1 \leq p < \infty$ $\forall \varepsilon > 0$, let $[f] \in L_p(E, \mathbb{R})$. Then $f \in \mathcal{L}_p(E, \mathbb{R})$ and we may write

$$f = f^+ - f^-,$$

where $f^+, f^- \in \mathcal{L}_p(E, \mathbb{R})$. By Proposition 30, we can find simple functions

$$0 \leq \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \ldots \leq f^+$$

such that

$$f^+(x) = \lim_{n \to \infty} \varphi_n(x), \quad x \in E.$$

Note that

$$\int_E |\varphi_n|^p \leq \int_E |f^+|^p \leq \int_E |f|^p < \infty,$$

and so $\varphi_n \in \mathrm{SIMP}_p(E, \mathbb{R})$, for $n \geq 1$. Thus, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n \to \infty} \int_E |f^+ - \varphi_n|^p = \int_E \lim_{n \to \infty} |f^+ - \varphi_n|^p = 0$$

Thus we can find some $N_1 > 0$, such that for $n > N_1$, we have

$$\|f^+ - \varphi_n\|_p < \frac{\varepsilon}{2}.$$

Similarly, we can find simple functions $\psi_1, \psi_2, \ldots \in \mathrm{SIMP}_P(E, \mathbb{R})$, such that

$$0 \leq \psi_1 \leq \psi_2 \leq \psi_3 \leq \ldots - f^-,$$

such that

$$f^-(x) = \lim_{n \to \infty} \psi_n(x), \quad x \in E,$$

and so that we can find $N_2 > 0$ where $\forall n > N_2$, we have

$$\left\| f^- - \psi_n \right\|_p < \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned} \| f - (\varphi_n + \psi_n) \|_p &= \left\| f^+ - f^- - \varphi_n - \psi_n \right\|_p \\ &\leq \left\| f^+ - \varphi_n \right\|_p + \left\| f^- - \psi_n \right\|_p \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Case 2: $p = \infty$ Let $\varepsilon > 0$, $[f] \in L_\infty(E, \mathbb{R})$, and $M = \|f\|_\infty$. Then range $f \subseteq [-M, M] =: I$. Now choose $N > 0$ such that $\frac{1}{N} < \varepsilon$. [2] Let

$$I_k = \left[-M + \frac{k}{N}, -M + \frac{k+1}{N} \right)$$

for $k \in \{0, \ldots, 2MN - 2\}$, and $I_{2MN} = \left[M - \frac{1}{N}, M \right]$.

Let $H_k := f^{-1}(I_k)$, for $k \in \{0, \ldots, 2MN - 1\}$. Then H_k is measurable by the measurability of f. Let

$$\varphi := \sum_{k=0}^{2MN-2} \left(-M + \frac{k}{N} \right) \chi_{H_k}.$$

It is clear that $\varphi \in \mathrm{SIMP}(E, \mathbb{R}) = \mathrm{SIMP}_\infty(E, \mathbb{R})$. Furthermore,

$$|f(x) - \varphi(x)| \leq \frac{1}{N} < \varepsilon \quad \forall x \in E.$$

It follows that

$$\|[f] - [\varphi]\|_\infty < \varepsilon.$$

This completes the proof. □

[2] Let us break I into intervals of length $\frac{1}{N}$. Doing this will allow $\left| f(x) - (-M + \frac{k}{N}) \chi_{f^{-1}(I_k)} \right| \leq \frac{1}{N}$.

Proposition 60 (Density of Equivalence Classes of Step Functions in L_p Spaces)

Let $a < b \in \mathbb{R}$. If $1 \leq p < \infty$, then

$$[\mathrm{STEP}([a, b], \mathbb{K})]$$

is dense in

$$(L_p([a,b],\mathbb{K}), \|\cdot\|_p).$$

Proof

By a similar argument to what we provided for the case of $\mathbb{K} = \mathbb{C}$, it suffices for us to show that the statement is true for the case when $\mathbb{K} = \mathbb{R}$.

Notice that $[a,b] \in \mathfrak{M}(\mathbb{R})$, and $m[a,b] = b - a < \infty$. Let us see for ourselves that $\mathcal{Y} := [\mathrm{SIMP}([a,b],\mathbb{R})]$ and $\mathcal{Z} := [\mathrm{STEP}([a,b],\mathbb{R})]$ are linear manifolds in $L_p([a,b],\mathbb{R})$. It is rather clear that $\mathcal{Y}, \mathcal{Z} \subseteq L_p([a,b],\mathbb{R})$. To show that $\mathcal{Y}$ is a linear manifold, we see that for $\varphi, \psi \in \mathrm{SIMP}([a,b],\mathbb{R})$ and $c \in \mathbb{R}$, if we suppose wlog that $N < M$ and define $E_n = \emptyset$ and $\alpha_n = 0$ for $N < n \leq M$, then

$$\begin{aligned} c\varphi + \psi &= c \sum_{n=1}^{N} \alpha_n \chi_{E_n} + \sum_{m=1}^{M} \beta_m \chi_{H_m} \\ &= \sum_{n=1}^{M} c(\alpha_n + \beta_n) \chi_{E_n \cup H_n} \in \mathrm{SIMP}([a,b],\mathbb{R}). \end{aligned}$$

To show that $\mathcal{Z}$ is a linear manifold, we see that for $\varphi, \psi \in \mathrm{STEP}([a,b],\mathbb{R})$ and $c \in \mathbb{R}$, if we suppose wlog that $N < M$ and define $I_n = \emptyset$ and $\alpha_n = 0$ for $N < n \leq M$, and define coefficients such that

$$c_n(x) = \begin{cases} a_n + b_n & x \in I_n \cap J_n \\ a_n & x \in I_n \setminus J_n \\ b_n & x \in J_n \setminus I_n \end{cases},$$

then

$$\begin{aligned} (c\varphi + \psi)(x) &= c \sum_{n=1}^{N} \alpha_n \chi_{I_n} + \sum_{m=1}^{M} \beta_m \chi_{J_m} \\ &= c \sum_{n=1}^{M} c_n(x) (\chi_{I_n \setminus J_n} + \chi_{I_n \cap J_n} + \chi_{J_n \setminus I_n})(x) \\ &\in \mathrm{STEP}([a,b],\mathbb{R}). \end{aligned}$$

From here, notice that by our warning on page 90, $\mathcal{Z} \subseteq \mathcal{Y}$. Furthermore, if we define

$$\mathcal{B} := \{\chi_H : H \in \mathfrak{M}([a,b])\},$$

then

$$\mathcal{Y} = \operatorname{span}\{[\varphi] : \varphi \in \mathcal{B}\},$$

and so along with Proposition 60, span $\mathcal{B}$ is dense in $(L_p([a,b], \mathbb{R}, \|\cdot\|_p))$. From Lemma 58, it suffices for us to show that $\mathcal{B} \subseteq \overline{\mathcal{Z}}$.[3] By Theorem 18, we can find an open $H \subseteq G \subseteq \mathbb{R}$ such that

$$m(G \setminus H) < \frac{\varepsilon}{2}.$$

We may write

$$G = \bigcup_{n=1}^{\infty} (a_n, b_n).$$

It is important that we note that each of the interval is finite, since $mH \leq m[a,b] < \infty$, and $m(G \setminus H) < \infty$, and thus $m(G) = m(H) + m(G \setminus H) < \infty$. Furthermore, some of the (a_n, b_n)'s may be empty sets.

Now let

$$G_k = \bigcup_{n=1}^{k} (a_n, b_n).$$

Clearly, $\lim_{k\to\infty} G_k = G$. Then we may choose $N > 0$ such that

$$m(G \setminus G_N) = \sum_{n=N+1}^{\infty} m([a_n, b_n]) < \frac{\varepsilon}{2}.$$

Let $\varphi = \chi_{G_N \cap [a,b]}$. It is clear that $\varphi \in \mathrm{STEP}([a,b], \mathbb{R})$.

It remains to show that

$$\nu_p(\chi_H - \varphi) = \int_{[a,b]} |\chi_H - \varphi|^p < \varepsilon.$$

Notice that

$$|\chi_H(x) - \varphi(x)| = \begin{cases} |1-0| = 1 & x \in H \setminus G_N \\ |0-1| = 1 & x \in (G_N \cap [a,b]) \setminus H \\ |1-1| = 0 & x \in H \cap G_N \\ |0-0| = 0 & x \notin H \cup G_N \end{cases}.$$

[3] We want to approximate any element $[\chi_H] \in \mathcal{B}$ using intervals. Realizing that we are in $\mathbb{R}$, we know that any open set $G \subseteq \mathbb{R}$ can be written as a disjoint union of open intervals. Furthermore, if we pick an open set G that closely encloses H, then we obtain disjoint open sets that closely approximates H.

It thus follows that

$$\begin{aligned}
\nu_p(\chi_H - \varphi) &= \int_{[a,b]} |\chi_H - \varphi|^p \\
&= \int_E |\chi_H - \varphi| \\
&= m(H \setminus G_N) + m((G_N \cap [a,b]) \setminus H) \\
&\leq m(G \setminus G_N) + m(G \setminus H) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

where $E = (H \setminus G_N) \cup ((G_N \cap [a,b]) \setminus H)$. It thus follows that $[\chi_H] \in \overline{\operatorname{span}} \mathcal{Z}$, and so $\mathcal{Z} = [\text{STEP}([a,b], \mathbb{R})]$ is dense in $(L_p([a,b], \mathbb{R}), \|\cdot\|_p)$. □

Note 13.1.1

Lemma 58 greatly simplified our proof above. We completely circumvented the need to pick an arbitrary element from $L_p([a,b], \mathbb{K})$ and try to approximate it using step functions. Instead, we need only approximate characteristic functions of measurable sets.

We shall use the same approach as we did in the proof above to show that the equivalence classes of continuous functions on a closed interval $[a,b]$, over $\mathbb{K}$, is dense in $(L_p([a,b], \mathbb{K}), \|\cdot\|_p)$.

Theorem 61 (Density of Equivalence Classes of Continuous Functions in L_p Spaces)

Let $a < b \in \mathbb{R}$. If $1 \leq p < \infty$, then $[\mathcal{C}([a,b], \mathbb{K})]$ is dense in $(L_p([a,b], \mathbb{K}), \|\cdot\|_p)$.

Proof

We may once again assume that $\mathbb{K} = \mathbb{R}$, as we did in the last 2 proofs.

Let

$$\mathcal{B} := \{[\chi_{[r,s]}] : a \leq r < s \leq b\}.$$

By Proposition 60, $\overline{\operatorname{span}}\mathcal{B} = L_p([a,b], \mathbb{R})$. Let

$$\mathcal{Z} := [\mathcal{C}([a,b], \mathbb{R})].$$

By Lemma 58, it suffices to show that $\mathcal{B} \subseteq \overline{\mathcal{Z}}$.

Let $\varepsilon > 0$ and $\chi_{[r,s]} \in [\chi_{[r,s]}] \in \mathcal{B}$. Let $\frac{s-r}{2} > \delta > 0$ so that we consider the function

$$f_\delta(x) = \begin{cases} 0 & x \in x \leq r \text{ or } x \geq s \\ \frac{1}{\delta}(x-r) & r < x \leq r+\delta \\ 1 & r+\delta < x < s-\delta \\ -\frac{1}{\delta}(x-s) & s-\delta \leq x < s \end{cases}.$$

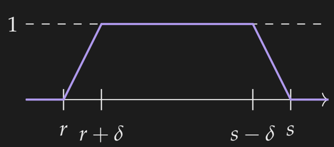

Figure 13.1: Shape of the continuous function f_δ for approximating $\chi_{[r,s]}$

Then

$$\begin{aligned} \left\| [\chi_{[r,s]}] - [f_\delta] \right\|_p^p &= \int_{[a,b]} \left| \chi_{[r,s]} - f_\delta \right|^p \\ &\leq \int_{[r,r+\delta] \cup [s-\delta,s]} 1^p \\ &= m([r, r+\delta]) + m([s-\delta, s]) \\ &= 2\delta. \end{aligned}$$

Then picking $\delta < \frac{\varepsilon}{2}$ in the first place, our work is done. □

Recall that a topological space is said to be **separable** if it admits a countable dense subset.

Exercise 13.1.2 (A way of finding a countable subset in a separable metric space)

Suppose (X, d) is a separable metric space, $\delta > 0$, and

$$Y := \{x_\lambda : \lambda \in \Lambda\} \subseteq X \text{ satisfies } d(x_\alpha, x_\beta) \geq \delta \text{ for all } \alpha \neq \beta \in \Lambda.$$

Then Λ is countable. [4]

[4] We may intuitively think of the flow of the proof as follows. If we can find such a Y whose elements are always δ away from one another in a separable metric space, then this Y should end up swallowing elements in X almost everywhere, and in particular, Y would be at least countable. However, Y is at most countable since it cannot be dense (elements that are within δ away from any element of Y cannot be closely approximated).

Corollary 62 (Separability of L_p Spaces)

Let $a < b \in \mathbb{R}$.

1. *If $1 \leq p < \infty$, then $(L_p([a,b], \mathbb{R}), \|\cdot\|_p)$ is separable.*
2. *If $p = \infty$, then $(L_\infty([a,b], \mathbb{K}), \|\cdot\|_\infty)$ is not separable.*

Proof

1. Fix $1 \leq p < \infty$. Recall from Remark 13.1.1 that for $[f], [g] \in L_\infty([a,b], \mathbb{K}) \subseteq L_p([a,b], \mathbb{K})$, we have

$$\|[f] - [g]\|_p \leq \|[f] - [g]\|_{\sup} \cdot m([a,b])^{\frac{1}{p}} = \|[f] - [g]\|_{\sup} (b-a)^{\frac{1}{p}}.$$

Let $\varepsilon > 0$ and $[h] \in L_p([a,b], \mathbb{K})$. By the density of $[\mathcal{C}([a,b], \mathbb{K})]$ in $L_p([a,b], \mathbb{K})$, we can find $[g] \in [\mathcal{C}([a,b], \mathbb{K})]$ such that

$$\|[h] - [g]\|_p < \frac{\varepsilon}{3}.$$

By the **Weierstrass Approximation Theorem**, we can find a polynomial $p(x) = p_0 + p_1 x + \ldots + p_m x^m \in \mathbb{C}[x]$ such that

$$\|[g] - [p]\|_\infty = \|g - p\|_{\sup} < \frac{\varepsilon}{3(b-a)^{\frac{1}{p}}}.$$

By the density of $\mathbb{Q}$ in $\mathbb{R}$, we can find a polynomial $q(x) = q_0 + q_1 x + \ldots + q_n x^n \in (\mathbb{Q} + i\mathbb{Q})[x]$ such that

$$\|[p] - [q]\|_\infty = \|p - q\|_{\sup} < \frac{\varepsilon}{3(b-a)^{\frac{1}{p}}}.$$

Observe that

$$\begin{aligned}
&\|[h] - [q]\|_p \\
&\leq \|[h] - [g]\|_p + \|[g] - [p]\|_p + \|[p] + [q]\|_p \\
&\leq \|[h] - [g]\|_p + \|[g] - [p]\|_\infty (b-a)^{\frac{1}{p}} + \|[p] - [q]\|_\infty (b-a)^{\frac{1}{p}} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3(b-a)^{\frac{1}{p}}} (b-a)^{\frac{1}{p}} + \frac{\varepsilon}{3(b-a)^{\frac{1}{p}}} (b-a)^{\frac{1}{p}}
\end{aligned}$$

$$= \varepsilon.$$

Thus, this q is the polynomial from a countable subset. Therefore, $[(\mathbb{Q} + i\mathbb{Q})[x]]$ is dense in $(L_p([a,b],\mathbb{K}), \|\cdot\|_p)$.

2. Consider $a \leq r_1 < s_1 \leq b$ and $a \leq r_2 < s_2 \leq b$, with $r_1 \neq r_2$ and $s_1 \neq s_2$. Then the **symmetric difference**

$$[r_1, s_1]\Delta[r_2, s_2] := ([r_1, s_1] \cup [r_2, s_2]) \setminus ([r_1, s_1] \cap [r_2, s_2])$$

contains an interval, say, $[u, v] \subseteq [a, b]$. Notice that for any $x \in [u, v]$,

$$\left|\chi_{[r_1,s_1]}(x) - \chi_{[r_2,s_2]}\right| = 1,$$

and so

$$\left\|[\chi_{[r_1,s_1]}] - [\chi_{[r_2,s_2]}]\right\|_\infty = \left\|[\chi_{[r_1,s_1]\Delta[r_2,s_2]}]\right\|_\infty = 1.$$

Consider $\Lambda := \{(r,s) \in \mathbb{R}^2 : a \leq r < s \leq b\}$. It is clear that Λ is uncountable. For any $(r_1, s_1) \neq (r_2, s_2) \in \Lambda$, by our above argument, we have

$$\left\|\chi_{[r_1,s_1]} - \chi_{[r_2,s_2]}\right\|_{\sup} = 1.$$

By Exercise 13.1.2, we have that $L_\infty([a,b],\mathbb{K})$ be must be separable. [5] □

[5] All the elements $\chi_{[r,s]}$ are 1-away from one another, and so the contrapositive of the exercise gives us this counterexample.

14 Lecture 14 Jun 25th 2019

14.1 Hilbert Spaces

Given $E \in \mathfrak{M}(\mathbb{R})$, we've seen that for $1 \leq p \leq \infty$, $(L_p(E, \mathbb{R}), \|\cdot\|_p)$ is a Banach space, i.e. a complete normed linear space. The case where $p = 2$ is a special space that merits our attention.

Definition 40 (Inner Product)

*An **inner product** on a $\mathbb{K}$-vector space $\mathcal{H}$ is a function*

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{K}$$

that satisfies

1. *(**positive definiteness**) $\langle x, x \rangle \geq 0$ for all $x \in \mathcal{H}$, and $\langle x, x \rangle = 0$ iff $x = 0$;*
2. *(**conjugate bilinear**) for all $w, x, y, z \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{K}$,*

$$\langle \alpha w + x, y + \beta z \rangle = \alpha \langle w, y \rangle + \langle x, y \rangle + \alpha \overline{\beta} \langle w, z \rangle + \overline{\beta} \langle x + z \rangle;$$

3. *(**conjugate symmetry**) for all $x, y \in \mathcal{H}$, $\langle x, y \rangle = \overline{\langle y, x \rangle}$.*

Definition 41 (Inner Product Space)

*An **inner product space (IPS)** is a vector space $\mathcal{H}$ endowed with an inner product.*

Definition 42 (Orthogonality)

*We say that x, y in an IPS $\mathcal{H}$ are **orthogonal** if*

$$\langle x, y \rangle = 0.$$

Theorem 63 (Cauchy-Schwarz Inequality)

Suppose $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is an IPS over $\mathbb{K}$. Then for all $x, y \in \mathcal{H}$,

$$|\langle x, y \rangle| \leq \langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}}.$$

Proof

Note that if $\langle x, y \rangle = 0$, then there is nothing to show, since inner products are positive definite. So suppose $\langle x, y \rangle \neq 0$. [1]

[1] This proof is said to be typical of any kind of Cauchy-Schwarz-like inequality. I am making this note because this is the rare time that I have actually seen one in pure mathematics (still a greenhorn with questionable basics).

Let $\kappa \in \mathbb{K}$. Notice that

$$\begin{aligned} 0 &\leq \langle x - \kappa y, x - \kappa y \rangle \\ &= \langle x, x \rangle - \kappa \langle y, x \rangle - \overline{\kappa} \langle x, y \rangle + |\kappa|^2 \langle y, y \rangle. \end{aligned}$$

So pick

$$\kappa = \frac{\langle x, y \rangle}{\langle y, y \rangle}.$$

Then we have

$$\begin{aligned} 0 &\leq \langle x, x \rangle - \frac{\langle x, y \rangle}{\langle y, y \rangle} \langle y, x \rangle - \frac{\langle y, x \rangle}{\langle y, y \rangle} \langle x, y \rangle + \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle^2} \langle y, y \rangle \\ &= \langle x, x \rangle - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} - \frac{|\langle x, y \rangle^2|}{\langle y, y \rangle} + \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} \\ &= \langle x, x \rangle - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle}. \end{aligned}$$

Thus

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle.$$

Hence

$$|\langle x, y\rangle| \leq \langle x, x\rangle^{\frac{1}{2}} \langle y, y\rangle^{\frac{1}{2}}. \qquad \square$$

Proposition 64 (Norm Induced by The Inner Product)

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be an IPS. Then the map

$$\|x\| := \langle x, x\rangle^{\frac{1}{2}}, \quad x \in \mathcal{H}$$

defines a norm on $\mathcal{H}$, called the ***norm induced by the inner product****.*

Proof

Positive Definiteness This immediately from the definition of an inner product.

Scalar Multiplication Let $\kappa \in \mathbb{K}$ and $x \in \mathcal{H}$. Then

$$\|\kappa x\|^2 = \langle \kappa x, \kappa x\rangle = |\kappa|^2 \langle x, x\rangle = |\kappa|^2 \|x\|^2.$$

Thus

$$\|\kappa x\| = |\kappa| \|x\|.$$

Triangle Inequality By the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&\leq \|x\|^2 + |\langle x, y\rangle| + |\langle y, x\rangle| + \|y\|^2 \\
&= \|x\|^2 + 2\langle x, x\rangle^{\frac{1}{2}} \langle y, y\rangle^{\frac{1}{2}} + \|y\|^2 \\
&= \|x\|^2 + 2\|x\| \|y\| + \|y\|^2 \\
&= (\|x\| + \|y\|)^2.
\end{aligned}$$

Hence

$$\|x + y\| \leq \|x\| + \|y\|. \qquad \square$$

It follows that every IPS $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is also a normed linear space (NLS). Furthermore, norms induce metrics, and so every IPS is also a metric space. Figure 14.1 is a highly abstract illustration of the idea.

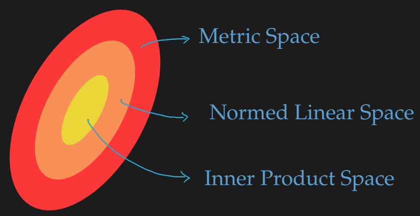

Figure 14.1: Hierarchy of Spaces, from Metric Space to Normed Linear Space, then down to Inner Product Space

Definition 43 (Hilbert Space)

A ***Hilbert space*** *is a complete IPS.*

Example 14.1.1

1. Let $N \geq 1$ be an integer, and $\mathcal{H} = \mathbb{C}^N$. For

$$x = (x_n)_{n=1}^N, \ y = (y_n)_{n=1}^N \in \mathbb{C}^N,$$

we define

$$\langle x, y \rangle = \sum_{n=1}^N x_n \overline{y_n}$$

as the inner product on $\mathbb{C}^N$. This is, in fact, called the **standard inner product** on $\mathbb{C}^N$. Furthermore, $\mathbb{C}^N$ is complete wrt to the norm induced by this inner product. Thus

$$(\mathbb{C}^N, \langle \cdot, \cdot \rangle)$$

is a Hilbert space.

2. We can make the above slightly more general. Fix $1 \leq N \in \mathbb{N}$, and choose some

$$\rho_1, \rho_2, \ldots, \rho_N \in \mathbb{R}^+.$$

Exercise 14.1.1

Check that $\mathbb{C}^N$, with the function

$$\langle x, y \rangle_\rho := \sum_{n=1}^N \rho_n x_n \overline{y_n}$$

that you are to check is an inner product, is a Hilbert space.

3. The following is a space that will be very important for us. The set

$$\ell_2(\mathbb{K}) := \{(x_n)_{n \in \mathbb{N}} : x_n \in \mathbb{K}, \sum |x_n|^2 < \infty\},$$

with the inner product

$$\langle (x_n)_n, (y_n)_n \rangle := \sum_{n=1}^{\infty} x_n \overline{y_n},$$

is called the **sequence space ℓ_2** with its **standard inner product**.
The space

$$(\ell_2(\mathbb{K}), \langle \cdot, \cdot \rangle)$$

is a Hilbert space. [2]

[2] We saw, in PMATH 351, that with the norm induced by this inner product, $\ell_2(\mathbb{K})$ is a complete metric space.

Let us now look at an inner product that we shall define on L_2.

Theorem 65 (The Standard Inner Product for $L_2(E, \mathbb{K})$)

Let $E \in \mathfrak{M}(\mathbb{R})$. The map

$$\begin{aligned} \langle \cdot, \cdot \rangle : L_2(E, \mathbb{K}) \times L_2(E, \mathbb{K}) &\to \mathbb{K} \\ ([f], [g]) &\mapsto \int_E f\overline{g} \end{aligned}$$

is an inner product on $L_2(E, \mathbb{K})$.

Furthermore, the norm induced by this inner product is the L_2-norm $\|\cdot\|_2$ on $L_2(E, \mathbb{K})$. Since $(L_2(E, \mathbb{K}), \|\cdot\|_2)$ is complete, $(L_2(E, \mathbb{K}), \langle \cdot, \cdot \rangle)$ is a Hilbert space.

Proof

Before anything else, we need to show that $\langle \cdot, \cdot \rangle$ is well-defined.
Notice that if $[f_1] = [f_2]$ and $[g_1] = [g_2]$ in $L_2(E, \mathbb{K})$, then $f_1\overline{g_1} = f_2\overline{g_2}$ a.e. on E, and so

$$\langle [f_1], [g_1] \rangle = \int_E f1\overline{g_1} = \int_E f_2\overline{g_2} = \langle [f_2], [g_2] \rangle.$$

Furthermore, by Theorem 49, we have that

$$\int_E |f\overline{g}| = \|[f\overline{g}]\|_1 \leq \|[f]\|_2 \, \|[\overline{g}]\|_2 < \infty.$$

Thus $\langle \cdot, \cdot \rangle$ is indeed well-defined.

Showing that $\langle \cdot, \cdot \rangle$ is an inner product **(Positive Definiteness)** Let $[f] \in L_2(E, \mathbb{K})$. Notice that

$$f\overline{f} = |f|^2 \geq 0.$$

Thus

$$\langle [f], [f] \rangle = \int_E f\overline{f} = \int_E |f|^2 \geq 0.$$

Now if $[f] = [0] \in L_2(E, \mathbb{K})$, then

$$\langle [f], [f] \rangle = \int_E |f|^2 = \int_E 0^2 = 0.$$

(Conjugate Bilinearity) Let $[f], [g], [h] \in L_2(E, \mathbb{K})$, and $\alpha, \beta \in \mathbb{K}$. Then

$$\begin{aligned}\langle \alpha[f] + \beta[g], [h] \rangle &= \int_E (\alpha f + \beta g)\overline{h} \\ &= \alpha \int_E f\overline{h} + \beta \int_E g\overline{h} \\ &= \alpha \langle [f], [h] \rangle + \beta \langle [g], [h] \rangle,\end{aligned}$$

and

$$\begin{aligned}\langle [f], \alpha[g] + \beta[h] \rangle &= \int_E f(\overline{\alpha g + \beta h}) \\ &= \overline{\alpha} \int_E f\overline{g} + \overline{\beta} \int_E f\overline{h} \\ &= \overline{\alpha} \langle [f], [g] \rangle + \overline{\beta} \langle [f], [h] \rangle.\end{aligned}$$

(Conjugate Symmetry) Let $[f], [g] \in L_2(E, \mathbb{K})$. Then

$$\langle [f], [g] \rangle = \int_E f\overline{g} = \overline{\int_E \overline{f\overline{g}}} = \overline{\int_E \overline{f} g} = \overline{\langle [g], [f] \rangle}.$$

Showing that the norm induced by $\langle \cdot, \cdot \rangle$ is the L_2-norm
By Proposition 64, we have for any $[f] \in L_2(E, \mathbb{K})$,

$$\| [f] \| := \langle [f], [f] \rangle^{\frac{1}{2}} = \left(\int_E |f|^2 \right)^{\frac{1}{2}} = \| [f] \|_2 . \qquad \square$$

Definition 44 (Orthonormality)

Let $\mathcal{E}$ be a subset of an IPS $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. We say that $x \in \mathcal{E}$ has ***norm 1*** *if*

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}} = 1.$$

We say that $x, y \in \mathcal{E}$ are ***orthonormal*** *if x, y each has norm 1, and they are orthogonal to one another, i.e. $\langle x, y \rangle = 0$.*

We say that $\mathcal{E}$ is an ***orthonormal set*** *if $\forall x, y \in \mathcal{E}$, x and y are orthonormal.*

Definition 45 (Orthonormal Basis)

Let $\mathcal{H}$ be a Hilbert space. An ***orthonormal basis (ONB)*** *(or* ***Hilbert space basis****) for $\mathcal{H}$ is a* ***maximal (wrt inclusion) orthonormal set*** *in $\mathcal{H}$.*

Remark 14.1.1

1. *By* ***Zorn's Lemma****, we can extend every orthonormal set in $\mathcal{H}$ to an ONB for $\mathcal{H}$.*
2. *If $\mathcal{H}$ is infinite-dimensional, then an ONB for $\mathcal{H}$ is* ***not*** *a* ***Hamel basis*** [3] *for $\mathcal{H}$.*

[3] The **Hamel basis** is a basis that we are rather familiar with, coming from a finite-dimensional world, where the span of an ONB is the entire space.

Example 14.1.2

1. Let $N \geq 1$ be an integer, and consider $\mathcal{H} = \mathbb{C}^N$ endowed with the standard inner product $\langle \cdot, \cdot \rangle$. For $1 \leq n \leq N$, define

$$e_n := (\delta_{n,k})_{k=1}^N,$$

   where $\delta_{a,b}$ denotes the **Kronecker delta function**. Then $\{e_n\}_{n=1}^N$ is an ONB for $\mathbb{C}^N$.

2. Let $1 \leq N \in \mathbb{N}$ and $\rho_k = k$, for $1 \leq k \leq N$. Set

$$e_n := \left(\frac{1}{\sqrt{k}} \delta_{n,k} \right)_{k=1}^N .$$

   Then $\{e_n\}_{n=1}^N$ is **also** an ONB for $\mathbb{C}^N$, with the rather awkward inner product from our last example:

$$\langle x, y \rangle_\rho = \sum_{n=1}^{N} \rho_n x_n \overline{y_n}.$$

3. Generalizing the first example here to infinite dimensions, let $\mathcal{H} = \ell_2(\mathbb{K})$, with its standard inner product. For $n \geq 1$, let

$$e_n = (\delta_{n,k})_{k=1}^{\infty}.$$

Exercise 14.1.2

Show that $\{e_n\}_{n=1}^{N}$ *is an ONB for* $\ell_2(\mathbb{K})$.

4. Now for an orthonormal basis that is highly relevant to us. Consider $\mathcal{H} = L_2([0, 2\pi], \mathbb{C})$, of which we have shown is a Hilbert space, with its standard inner product

$$\langle [f], [g] \rangle := \int_E f\overline{g}.$$

For $n \in \mathbb{Z}$, define the continuous function

$$\begin{aligned} \xi_n : [0, 2\pi] &\to \mathbb{C} \\ \theta &\mapsto \frac{1}{\sqrt{2\pi}} e^{in\theta}. \end{aligned}$$

Then $[\xi_n] \in L_2([0, 2\pi], \mathbb{C})$ for all $n \in \mathbb{Z}$. In A4, we shall see that $\{[\xi_n]\}_{n \in \mathbb{Z}}$ is an ONB for $L_2([0, 2\pi], \mathbb{C})$.

We recall the following result from linear algebra:

Theorem 66 (Gram-Schmidt Orthogonalisation Process)

If $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ *is a Hilbert space over* $\mathbb{K}$*, and* $\{x_n\}_{n=1}^{\infty}$ *is a* ***linearly independent set*** *in* $\mathcal{H}$*, then we can find an orthonormal set* $\{y_n\}_{n=1}^{\infty}$ *in* $\mathcal{H}$ *so that*

$$\operatorname{span}\{x_1, \ldots, x_N\} = \operatorname{span}\{y_1, \ldots, y_N\}$$

for all $N \geq 1$.

Proof

First, set

$$y_1 = \frac{x_1}{\|x_1\|}.$$

Notice that

$$\begin{aligned}\langle x_2 - \langle x_2, y_1\rangle y_1, y_1\rangle &= \langle x_2, y_1\rangle - \langle x_2, y_1\rangle\langle y_1, y_1\rangle \\ &= \langle x_2, y_1\rangle - \langle x_2, y_1\rangle \cdot 1 = 0.\end{aligned}$$

To get norm 1, we can then set

$$y_2 = \frac{x_2 - \langle x_2, y_1\rangle y_1}{\|x_2 - \langle x_2, y_1\rangle y_1\|}.$$

By induction, one can show that

$$y_N = \frac{x_N - \sum_{n=1}^{N-1}\langle x_N, y_n\rangle y_n}{\left\| x_N - \sum_{n=1}^{N-1}\langle x_N, y_n\rangle y_n \right\|},$$

for $N \geq 1$, works. □

Theorem 67 (The Pythagorean Theorem and Parallelogram Law)

Let $\mathcal{H}$ be a Hilbert space and suppose that $x_1, x_2, \ldots, x_n \in \mathcal{H}$.

1. *(**The Pythagorean Theorem**) If $\{x_n\}_{n=1}^N$ is orthogonal, then*

$$\left\| \sum_{n=1}^N x_n \right\|^2 = \sum_{n=1}^N \|x_2\|^2.$$

2. *(**The Parallelogram Law**) We have*

$$\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 = 2\left(\|x_1\|^2 + \|x_2\|^2\right).$$

Proof

1. Since $\langle x_n, x_m \rangle = 0$ for all $n \neq m$, we have

$$\left\| \sum_{n=1}^{N} x_n \right\|^2 = \left\langle \sum_{n=1}^{N} x_n, \sum_{n=1}^{N} x_n \right\rangle = \sum_{n=1}^{N} \langle x_n, x_n \rangle = \sum_{n=1}^{N} \|x_n\|^2 .$$

2. We see that

$$\begin{aligned} \|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 &= \langle x_1 + x_2, x_1 + x_2 \rangle + \langle x_1 - x_2, x_1 - x_2 \rangle \\ &= 2\langle x_1, x_1 \rangle + 2\langle x_2, x_2 \rangle \\ &= 2 \left(\|x_1\|^2 + \|x_2\|^2 \right) . \end{aligned}$$

□

15 Lecture 15 Jun 27th 2019

15.1 Hilbert Spaces (Continued)

Theorem 68 (Closest Point from a Convex Set in a Hilbert Space)

Let $\mathcal{H}$ be a Hilbert space, and $K \subseteq \mathcal{H}$ be a closed, non-empty convex subset of $\mathcal{H}$. Given $x \in \mathcal{H}$, there exists a unique point $y \in K$ that is closest to x, i.e.

$$\|x - y\| = \operatorname{dist}(x, K) := \min\{\|x - z\| : z \in K\}.$$

The proof of Theorem 68 is left to the assignments.

Theorem 69 (A Way to Orthogonality)

Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{M} \subseteq \mathcal{H}$ be a closed subspace. Let $x \in \mathcal{H}$, and $m \in \mathcal{M}$. TFAE:

1. $\|x - m\| = \operatorname{dist}(x, \mathcal{M})$;
2. *The vector $x - m$ is orthogonal to $\mathcal{M}$, i.e.*

$$\langle x - m, y \rangle = 0 \text{ for all } y \in \mathcal{M}.$$

Proof

(1) $\implies$ (2) Suppose to the contrary that $\exists y \in \mathcal{M}$ such that

$$\kappa := \langle x - m, y \rangle \neq 0.$$

Wlog, suppose $\|y\| = 1$.[1] Consider $z := m + \kappa y \in \mathcal{M}$. Then

$$\begin{aligned} \|x - z\|^2 &= \langle x - z, x - z \rangle = \langle x - m - \kappa y, x - m - \kappa y \rangle \\ &= \|x - m\|^2 - \kappa \langle y, x - m \rangle - \overline{\kappa} \langle x - m, y \rangle + \kappa \overline{\kappa} \langle y, y \rangle \\ &= \|x - m\|^2 - \kappa \overline{\kappa} - \overline{\kappa} \kappa + \kappa \overline{\kappa} \\ &= \|x - m\|^2 - |\kappa|^2 < \|x - m\|^2, \end{aligned}$$

a contradiction. Thus, such a y cannot exist, and so the result holds.

(2) $\implies$ (1) Suppose $\forall y \in \mathcal{M}$, $\langle x - m, y \rangle = 0$. Write

$$\mathcal{M} \ni y = m + (y - m).$$

Observe that by the Pythagorean theorem,

$$\begin{aligned} \|x - y\| = \|x - m - y + m\| &= \|x - m\| + \|m - y\| \\ &\geq \|x - m\| \end{aligned}$$

since $\|m - y\| \geq 0$. Thus $\|x - m\| = \operatorname{dist}(x, \mathcal{M})$. □

[1] We may assume so since if $\|y\| \neq 1$, then we simply divide κ by $\|y\|$ and we'll get a y' with norm 1.

Let's have a little talk about **complements**.

Definition 46 (Perpendicular Space)

*Given any non-empty subset $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, we define the **perpendicular space** of S as*

$$\mathcal{S}^{\perp} := \{y \in \mathcal{H} : \langle x, y \rangle = 0, \ x \in \mathcal{S}\}.$$

Exercise 15.1.1

Show that $\mathcal{S}^{\perp}$ is a norm-closed subspace[2] *of $\mathcal{H}$.*

[2] A **norm-closed subspace** is a subspace that is closed under the norm of the ambient space.

Remark 15.1.1

1. *Observe that* $0 \in \mathcal{S}^\perp$ *always, and* $(\mathcal{S}^\perp)^\perp \supseteq \mathcal{S}$*. It thus follows that*

$$\left(\mathcal{S}^\perp\right)^\perp \supseteq \overline{\operatorname{span}}\mathcal{S},$$

the ***norm closure of the linear span*** *of* $\mathcal{S}$*.*

2. *Let* $\mathcal{V}$ *is a vector space and* $\mathcal{W}$ *is a (vector) subspace of* $\mathcal{V}$*. Let*

$$\{w_\lambda : \lambda \in \Lambda\}$$

be a (Hamel) basis for $\mathcal{W}$*. We may then extend* $\{w_\lambda : \lambda \in \Lambda\}$ *to be a basis of* $\mathcal{V}$*, such as*

$$\{w_\lambda : \lambda \in \Lambda\} \cup \{x_\gamma : \gamma \in \Gamma\}.$$

Let

$$\mathcal{X} := \operatorname{span}\{x_\gamma : \gamma \in \Gamma\}.$$

Then $\mathcal{X} \subseteq \mathcal{V}$ *is a subspace, and*

(a) $\mathcal{W} \cap \mathcal{X} = \{0\}$*; and*

(b) $\mathcal{V} = \mathcal{W} + \mathcal{X} := \{w + x : w \in \mathcal{W}, x \in \mathcal{X}\}$*.*

We say that $\mathcal{W}$ *is* ***alagebraically complemented*** *by* $\mathcal{X}$*. This existence of* $\mathcal{X}$ *says that* ***every subspace is algebraically complemented****.*

Note that $\mathcal{X}$ *is* ***not unique****. Indeed, if vectors of the basis for* $\mathcal{X}$ *are not of norm 1, then normalizing them all gives us an ONB for* $\mathcal{X}$*.*

We can do something similar with normed linear spaces (NLSs). If $\mathfrak{X}$ *is a Banach space and* $\mathfrak{Y}$ *is a closed subspace of* $\mathfrak{X}$*, we say that* $\mathfrak{Y}$ *is* ***topologically complemented*** *if there exists a* ***closed*** *subspace* $\mathfrak{Z} \subseteq \mathfrak{X}$ *such that* $\mathfrak{Z}$ *is an algebraic complement to* $\mathfrak{Y}$*, i.e. that*

(a) $\mathfrak{Y} \cap \mathfrak{Z} = \{0\}$*; and*

(b) $\mathfrak{X} = \mathfrak{Y} + \mathfrak{Z}$*.*

However, ***not all*** *closed subspace of a Banach space is topologically complemented.*

We shall write $\mathfrak{X} = \mathfrak{Y} \oplus \mathfrak{Z}$ *if* $\mathfrak{Z}$ *is a topological complement to* $\mathfrak{Y}$*.*

Now let $\mathcal{H}$ *be a Hilbert space and* $\mathcal{M} \subseteq \mathcal{H}$ *be a closed subspace.*

Claim: $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ *From* Exercise 15.1.1*,* $\mathcal{M}^\perp$ *is closed. Notice that*

Culture (Phillip's Theorem)

$c_0 = \{(x_n)_n \in \mathbb{K}^{\mathbb{N}} : \lim_{n\to\infty} x_n = 0\} \subseteq \ell_\infty$ *is not topologically complemented.*

Cited from Whitley, 1996.

if $z \in \mathcal{M} \cap \mathcal{M}^\perp$, then

$$\langle z, z \rangle = 0,$$

and so $z = 0$. Thus $\mathcal{M} \cap \mathcal{M}^\perp = \{0\}$.

Let $x \in \mathcal{H}$. By Theorem 68, $\exists m_1 \in \mathcal{M}$ such that

$$\|x - m_1\| = \operatorname{dist}(x, \mathcal{M}).$$

Furthermore, by Theorem 69, $m_2 := x - m_1 \in \mathcal{M}^\perp$. Thus we see that

$$x = m_1 + m_2 \in \mathcal{M} + \mathcal{M}^\perp.$$

Since $\mathcal{M}$ and $\mathcal{M}^\perp$ are both closed subspaces, we have $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. $\dashv$

*In fact, the above claim is much stronger than what immediately meets the eye. Given a Banach space $\mathfrak{X}$ and a topologically complemented closed subspace $\mathfrak{Y}$, there is generally no expectation of a **unique** topological complement for $\mathfrak{Y}$. For instance, $\mathfrak{X} = \mathbb{R}^2$ with, say, $\|\cdot\|_\infty$, if we let $\mathfrak{Y}$ be the x-axis, then any line that passes through the origin and not equal to the x-axis would be a closed subspace and is a topological complement to $\mathfrak{Y}$. However, in the above claim, the space $\mathcal{M}^\perp$ is **unique**, and we call $\mathcal{M}^\perp$ the **orthogonal complement** of $\mathcal{M}$.*

3. *The orthogonal projection* With $\mathcal{H}$ and $\mathcal{M}$ as in the last remark, we have that $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. Now for any $x \in \mathcal{H}$, if we suppose that we can write

$$m_1 + n_1 = x = m_2 + n_2,$$

where $m_1, m_2 \in \mathcal{M}$ and $n_1, n_2 \in \mathcal{M}^\perp$, then

$$0 = x - x = m_1 - m_2 + n_1 - n_2 \implies m_1 - m_2 = n_1 - n_2.$$

But $m_1 - m_2 \in \mathcal{M}$ and $n_1 - n_2 \in \mathcal{M}^\perp$, and so $m_1 - m_2 = 0 = n_1 - n_2$, i.e. $m_1 = m_2$ and $n_1 = n_2$. Thus, we may uniquely represent each $x \in \mathcal{H}$ as

$$x = m + n \quad \text{where } m \in \mathcal{M},\ n \in \mathcal{M}^\perp.$$

Now consider the map

$$P : \mathcal{H} \to \mathcal{M} \oplus \mathcal{M}^\perp$$

$$x \mapsto m.$$

This map P is called an ***orthogonal projection****.*

Continuity of the orthogonal projection *Observe that given $x_1 = m_1 + n_1, x_2 = m_2 + n_2 \in \mathcal{H}$ and $\kappa \in \mathbb{K}$, we have*

$$\begin{aligned} P(\kappa x_1 + x_2) &= P(\kappa(m_1 + n_1) + m_2 + n_2) \\ &= \kappa m_1 + m_2 = \kappa P(x_1) + P(x_2). \end{aligned}$$

Thus P is linear. Furthermore,

$$P(P(x_1)) = P(m_1) = m_1.$$

Thus $P^2 = P$, and we say that P is an ***idempotent****.*

On a related note to the orthogonal projection, observe that the 'projection in the other way' is also an orthogonal projection. That is, $Q = I - P$, where I is the identity function, that would give $Q(x_1) = (I - P)(x_1) = m_1 + n_1 - m_1 = n_1$, is also an orthogonal projection.

In fact, for $x \in \mathcal{H}$, we have that

$$\|Px\|^2 = \|m\|^2 \leq \|m\|^2 + \|n\|^2 = \|m + n\|^2 = \|x\|^2.$$

Thus the operator norm on P is

$$\|P\| = \sup\{\|Px\| : \|x\| \leq 1\} \leq 1.$$

It follows that P is bounded. Since it is linear, it is also continuous.

Finally, notice that if $m \in \mathcal{M} \neq \{0\}$ such that $\|m\| = 1$, then

$$\|Pm\| = \|m\| = 1.$$

4. *Let $\emptyset \neq \mathcal{S} \subseteq \mathcal{H}$. By the first remark, if we let $\mathcal{M} = \overline{\operatorname{span}} \mathcal{S}$, then $\mathcal{M}$ is a closed subspace of $\mathcal{H}$. By the second remark, we have*

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp.$$

Exercise 15.1.2

Show that $\mathcal{S}^\perp = \mathcal{M}^\perp$.

Suppose $\exists 0 \neq x \in \left(\mathcal{S}^\perp\right)^\perp$ such that $x \notin \mathcal{M}$. Notice that since $x \in \mathcal{H}$, *we can write*

$$x = m_1 + m_2,$$

where $m_1 \in \mathcal{M}$ *and* $m_2 \in \mathcal{M}^\perp$. *Notice that* $m_2 \neq 0 \in \mathcal{M}^\perp = \mathcal{S}^\perp$, *since otherwise,* $x \in \mathcal{M}$. *But then*

$$\langle x, m_2 \rangle = \langle m_1 + m_2, m_2 \rangle = 0 + \|m_2\|^2 \neq 0.$$

Thus $x \notin \left(\mathcal{S}^\perp\right)^\perp$, *a contradiction. It follows that* $\left(\mathcal{S}^\perp\right)^\perp \subseteq \overline{\operatorname{span}}\mathcal{S}$, *and so by the first remark,*

$$\left(\mathcal{S}^\perp\right)^\perp = \overline{\operatorname{span}}\mathcal{S}.$$

Lemma 70 (Finite Dimensional Linear Manifolds are Norm-closed Subspaces)

Let $\mathcal{H}$ *be a Hilbert space over* $\mathbb{K}$, *and suppose that* $\mathcal{M} \subseteq \mathcal{H}$ *is a finite-dimensional linear manifold in* $\mathcal{H}$. *Then* $\mathcal{M}$ *is norm-closed, and hence a subspace of* $\mathcal{H}$.

Proof

The proof of Lemma 70 is left to the assignments. □

Proposition 71 (Formulae for Orthogonal Projections in Hilbert Spaces onto a Finite-Dimensional Subspace)

Suppose $\mathcal{M}$ *is a finite-dimensional subspace of a Hilbert space* $\mathcal{H}$ *over* $\mathbb{K}$. *Suppose that* $\exists N \in \mathbb{N} \setminus \{0\}$, *such that* $\{e_1, \ldots, e_N\}$ *is an ONB for* $\mathcal{H}$. *If* P *is the orthogonal projection of* $\mathcal{H}$ *onto* $\mathcal{M}$, *then*

$$Px = \sum_{n=1}^{N} \langle x, e_n \rangle e_n, \quad x \in \mathcal{H}.$$

Proof

Let $x \in \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, and write $x = m_1 + m_2$, with $m_1 \in \mathcal{M}$ and $m_2 \in \mathcal{M}^\perp$. By the second point in the last remark, we have that $Px = m_1$ is unique such that $x - Px \in \mathcal{M}^\perp$. Consider $w = \sum_{n=1}^N \langle x, e_n \rangle e_n$. For $m \in \{1, \ldots, N\}$, we observe that

$$\begin{aligned} \langle x - w, e_m \rangle &= \langle x, e_m \rangle - \sum_{n=1}^N \langle x, e_n \rangle \langle e_n, e_m \rangle \\ &= \langle x, e_m \rangle - \langle x, e_m \rangle \cancelto{1}{\langle e_m, e_m \rangle} \\ &= 0. \end{aligned}$$

Thus $x - w \in \mathcal{M}^\perp$, and so

$$Px = w = \sum_{n=1}^N \langle x, e_n \rangle e_n. \qquad \square$$

Theorem 72 (Bessel's Inequality)

If $\{e_n\}_{n=1}^\infty$ is an orthonormal set in a Hilbert space $\mathcal{H}$, then for each $x \in \mathcal{H}$,

$$\sum_{n=1}^\infty \left| \langle x, e_n \rangle \right|^2 \leq \left\| x \right\|^2.$$

Proof

For each $N \in \mathbb{N} \setminus \{0\}$, set

$$\mathcal{M}_N := \operatorname{span}\{e_1, \ldots, e_N\}.$$

Then each $\mathcal{M}_N$ is a finite-dimensional subspace of $\mathcal{H}$ with ONB $\{e_1, \ldots, e_N\}$.

For each N, let P_N be the orthogonal projection from $\mathcal{H}$ to $\mathcal{M}_N$. From the last discussion on the 3rd point of the last remark, since $\|e_n\| = 1$, we have that $\|P_N\| = 1$ for all N.

By Proposition 71, we observe that

$$\|x\|^2 \geq \|P_N x\|^2 = \left\| \sum_{n=1}^N \langle x, e_n \rangle e_n \right\|^2$$

$$
\begin{aligned}
&= \sum_{n=1}^{N} \left| \langle x, e_n \rangle e_n \right|^2 \\
&= \sum_{n=1}^{N} \left| \langle x, e_n \rangle \right|^2
\end{aligned}
$$

by the Pythagorean Theorem. □

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16.1 Hilbert Spaces (Continued 2)

Theorem 73 (Countability of an Orthonormal Set in a Separable Hilbert Space)

Let $\mathcal{H}$ be a(n) (infinite-dimensional) separable Hilbert space, and suppse that $\mathcal{E} \subseteq \mathcal{H}$ is an orthonormal set. Then $\mathcal{E}$ is countable, say as $\mathcal{E} = \{e_n\}_{n=1}^{\infty}$, and if $x \in \mathcal{H}$, then

$$\sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$$

converges in $\mathcal{H}$.

Proof

First, notice that for $x \neq y \in \mathcal{E}$, we have

$$\|x - y\| = \langle x - y, x - y \rangle^{\frac{1}{2}} = \left(\|x\|^2 + \|y\|^2 \right)^{\frac{1}{2}} = \sqrt{2}.$$

By Exercise 13.1.2, we have that $\mathcal{E}$ is indeed countable.

Let $x \in \mathcal{H}$ and $\varepsilon > 0$. For each $N \geq 1$, set

$$y_N = \sum_{n=1}^{N} \langle x, e_n \rangle e_n.^{1}$$

Since $\mathcal{H}$ is complete (for it is a Hilbert space), it suffices for us to show that $\{y_N\}_{N=1}^{\infty}$ is Cauchy.

[1] The keen reader should notice that we are simply taking $y_N = P_N x$ from the proof of Bessel's Inequality.

From Bessel's Inequality, we know that

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|^2 < \infty.$$

Thus, for any $\varepsilon > 0$, rearranging if necessary, we can find some $N_0 > 0$ such that

$$\sum_{n=N_0+1}^{\infty} |\langle x, e_n \rangle|^2 < \varepsilon.$$

Then for $M > N > N_0$, we see that

$$\begin{aligned} \|y_M - y_N\|^2 &= \left\| \sum_{n=1}^{M} \langle x, e_n \rangle e_n - \sum_{n=1}^{N} \langle x, e_n \rangle e_n \right\|^2 \\ &= \left\| \sum_{n=N+1}^{M} \langle x, e_n \rangle e_n \right\|^2 \\ &= \sum_{n=N+1}^{M} |\langle x, e_n \rangle|^2 \quad \because \text{Pythagorean Theorem} \\ &\leq \sum_{n=N_0}^{\infty} |\langle x, e_n \rangle|^2 < \varepsilon. \end{aligned}$$

It follows that the limit of the Cauchy sequence $\{y_N\}_{N=1}^{\infty}$ is in $\mathcal{H}$. □

Theorem 74 (Characterization of an ONB)

Let $\mathcal{E} = \{e_n\}_{n=1}^{\infty}$ be an orthonormal set in an infinite-dimensional, separable Hilbert space $\mathcal{H}$. TFAE:

1. *$\mathcal{E}$ is an ONB, i.e. $\mathcal{E}$ is a maximal orthonormal set in $\mathcal{H}$.*
2. $\overline{\operatorname{span}}\mathcal{E} = \mathcal{H}$.
3. *$\forall x \in \mathcal{H}$, $x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$.*
4. *(**Parseval's Identity**) $\forall x \in \mathcal{H}$, $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2$.*

Proof

(1) $\implies$ (2) Firstly, it is clear that $\mathcal{E} \subseteq \mathcal{M} := \overline{\operatorname{span}}\mathcal{E} \subseteq \mathcal{H}$. In particular, $\{0\} \neq \mathcal{E} \subseteq \mathcal{M}$, so $\mathcal{M}^{\perp} \neq \{0\}$. Then $\exists 0 \neq x \in \mathcal{M}^{\perp}$ such

that $\mathcal{E} \cup \{x\}$ is also an orthonormal basis, contradicting maximality of $\mathcal{E}$.

(2) $\implies$ (3) Let $\mathcal{M} = \overline{\operatorname{span}} \mathcal{E} = \mathcal{H}$. Let $y = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$. Observe that by a similar step in the proof in Proposition 71, we observe that

$$\langle x - y, e_m \rangle = 0, \quad \text{for each } m \in \mathbb{N} \setminus \{0\}.$$

It follows that $x - y \in \mathcal{M}^{\perp} = \{0\}$, and so $x = y$.

(3) $\implies$ (4) We see that

$$\begin{aligned}
\|x\|^2 &= \left\| \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n \right\|^2 \\
&= \langle \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n, \sum_{m=1}^{\infty} \langle x, e_m \rangle e_m \rangle \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle x, e_n \rangle \overline{\langle x, e_m \rangle} \langle e_n, e_m \rangle \\
&= \sum_{n=1}^{\infty} \langle x, e_n \rangle \overline{\langle x, e_n \rangle} \cancelto{1}{\langle e_n, e_n \rangle} \\
&= \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 .
\end{aligned}$$

(4) $\implies$ (1) Suppose $x \in \mathcal{E}^{\perp}$. Then for all $n \in \mathbb{N} \setminus \{0\}$,

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 = 0,$$

and so $x = 0$, i.e. $\mathcal{E}^{\perp} = \{0\}$. Hence $\mathcal{E}$ is indeed maximum. □

Definition 47 (Unitary Operator)

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces over $\mathbb{K}$. A map $U : \mathcal{H}_1 - > \mathcal{H}_2$ is called a ***unitary operator*** *if it is a linear bijection such that*

$$\langle Ux, Uy \rangle = \langle x, y \rangle$$

for all $x, y \in \mathcal{H}_1$.

Definition 48 (Isomorphism of Hilbert Spaces)

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces over $\mathbb{K}$. We say that $\mathcal{H}_1$ and $\mathcal{H}_2$ are ***isomorphic*** *if there exists a unitary operator $U : \mathcal{H}_1 -> \mathcal{H}_2$. We denote this relationship as $\mathcal{H}_1 \simeq \mathcal{H}_2$.*

Note 16.1.1

Note that $\forall x \in \mathcal{H}$, we have that

$$\|Ux\|^2 = \langle Ux, Ux \rangle = \langle x, x \rangle = \|x\|^2 .$$

In particular, unitary operators are ***isometries****. Furthermore, observe that*

$$\|U\| = \sup\{\|Ux\| : \|x\| \leq 1\} \leq 1,$$

and so unitary operators are bounded and continuous. Moreover, the inverse map $U^{-1} : \mathcal{H}_2 \to \mathcal{H}_1$ defined by $U^{-1}(Ux) := x$ is also linear, and

$$\langle U^{-1}(Ux), U^{-1}(Uy) \rangle = \langle x, y \rangle = \langle Ux, Uy \rangle,$$

the inverse of a unitary operator is also a unitary operator.

Remark 16.1.1

Note that if $\mathcal{L} \subseteq \mathcal{H}_1$ is a closed subspace, then $\mathcal{L}$ is complete, whence $U\mathcal{L}$ is also complete, and hence closed in $\mathcal{H}_2$.

The proof of the following theorem is left to the assignments.

Theorem 75 (Isomorphism of Infinite-dimensional Separable Hilbert Spaces)

Any 2 infinite-dimensional separable Hilbert spaces over $\mathbb{K}$ are isomorphic.

16.2 Introduction to Fourier Analysis

Remark 16.2.1

As a result of Theorem 75, it follows that if $\mathcal{H}$ is a complex, separable, infinite-dimensional Hilbert space, then $\mathcal{H} \simeq \ell_2$.

One must wonder why do we focus on $E = [-\pi, \pi]$. For a relatively good motivation for the things that are to come, please read Appendix A.

Now,

- *from Corollary 62, $L_2([-\pi, \pi], \mathbb{K})$ is separable;*
- *from Item 4 of Example 14.1.2, $L_2([-\pi, \pi], \mathbb{K})$ is infinite-dimensional, with the ONB $\{[\xi_n]\}_{n \in \mathbb{Z}}$; and*
- *by Theorem 65, $L_2([-\pi, \pi], \mathbb{K})$ is a Hilbert space, with the inner product*
$$\langle [f], [g] \rangle = \int_E f\overline{g}.$$

Let us define

$$\mathcal{L}_2(\mathbb{T}, \mathbb{C}) := \{ f : \mathbb{R} -> \mathbb{C} : f \text{ is measurable, } 2\pi\text{-periodic,} \\ \text{and } \int_{[\pi,\pi)} |f|^2 < \infty \bigg\}.$$

Exercise 16.2.1

Show that $\mathcal{L}_2(\mathbb{T}, \mathbb{C})$ is a vector space, and that the function

$$\begin{aligned} \nu_2 : \mathcal{L}_2(\mathbb{T}, \mathbb{C}) &\to \mathbb{R} \\ f &\mapsto \left(\frac{1}{2\pi} \int_{[-\pi,\pi)} |f|^2 \right)^{1/2} \end{aligned}$$

is a semi norm on $\mathcal{L}_2(\mathbb{T}, \mathbb{C})$.

Now let
$$\mathcal{N}_2(\mathbb{T}, \mathbb{C}) := \{ f \in \mathcal{L}_2(, \mathbb{T}, \mathbb{C}) : \nu_2(f) = 0 \}.$$

It follows that if $L_2(\mathbb{T}, \mathbb{C}) = \mathcal{L}_2(\mathbb{T}, \mathbb{C}) / \mathcal{N}(\mathbb{T}, \mathbb{C})$, then $[f] = [g] \in L_2(\mathbb{T}, \mathbb{C})$ iff $f = g$ a.e. on $\mathbb{R}$, or equivalently $f = g$ a.e. on $[-\pi, \pi)$, since they are 2π-periodic functions on $\mathbb{R}$. We can then obtain a norm on $L_2(\mathbb{T}, \mathbb{C})$ by setting $\| [f] \|_2 := \nu_2(f)$.

Furthermore, the function

$$\langle \cdot, \cdot \rangle_{\mathbb{T}} : L_2(\mathbb{T}, \mathbb{C}) \times L_2(\mathbb{T}, \mathbb{C}) \to \mathbb{C}$$
$$([f], [g]) \mapsto \frac{1}{2\pi} \int_{[-\pi,\pi)} f\overline{g}$$

is an inner product on $L_2(\mathbb{T}, \mathbb{C})$*, and* $\|\cdot\|_2$ *is precisely the norm induced by the inner product. By what we've seen in the last section,* $L_2(\mathbb{T}, \mathbb{C})$ *is complete wrt the norm* $\|\cdot\|$*, and is therefore a Hilbert space. One can finally verify that* $\{[\xi_n]\}_{n\in\mathbb{Z}}$*, where* $\xi_n(\varepsilon) = e^{in\theta}$*, is indeed and ONB for* $L_2(\mathbb{T}, \mathbb{C})$*.* 💬

Example 16.2.1 (Fourier Series for $L_2(\mathbb{T}, \mathbb{C})$, and the isomorphism between $L_2(\mathbb{T}, \mathbb{C})$ and $\ell_2(\mathbb{Z}, \mathbb{C})$)

Let $[f] \in L_2([-\pi, \pi], \mathbb{C})$. From A5Q4, we can show that $\{[\xi_n]\}_{n\in\mathbb{Z}}$, where

$$\xi_n : [-\pi, \pi] \to \mathbb{C}$$
$$\theta \mapsto \frac{1}{\sqrt{2\pi}} e^{in\theta},$$

is an ONB for $L_2([\pi, \pi], \mathbb{C})$. For any $n \in \mathbb{Z}$, let

$$\alpha_n^{[f]} := \langle [f], [\xi_n] \rangle.$$

We shall refer to $\alpha_n^{[f]}$ as the **n^{th}-Fourier coefficient** of $[f]$ relative to the ONB $\{[\xi_n]\}_{n\in\mathbb{Z}}$. By A7Q2, we have that the map

$$U : L_2(\mathbb{T}, \mathbb{C}) \to \ell_2(\mathbb{Z}, \mathbb{C})$$
$$[f] \mapsto \left(\alpha_n^{[f]}\right)_{n\in\mathbb{Z}}$$

is a unitary operator from the Hilbert space $L_2(\mathbb{T}, \mathbb{C})$ to $\ell_2(\mathbb{Z}, \mathbb{C})$.

In particular, U is injective. This means that if $[f], , [g] \in L_2(\mathbb{T}, \mathbb{C})$ and $\alpha_n^{[f]} = \alpha_n^{[g]}$ for all $n \in \mathbb{Z}$, then $f = g$ a.e. on $\mathbb{R}$. In other words, an element $[f] \in L_2(\mathbb{T}, \mathbb{C})$ is **completely determined by its Fourier coefficients**. Moreover, given any sequence $(\beta_n)_{n\in\mathbb{Z}} \in \ell 2(\mathbb{Z}, \mathbb{C})$, $\exists [f] \in L_2(\mathbb{T}, \mathbb{C})$ such that $\alpha_n^{[f]} = \beta_n$, for all $n \in \mathbb{Z}$.

Now let $[f] \in L_2(\mathbb{T}, \mathbb{C})$. For each $N \in \mathbb{N} \setminus \{0\}$, set

$$\Delta_N([f]) = \sum_{n=-N}^{N} \alpha_n^{[f]} [\xi_n].$$

We shall call $\Delta_N([f])$ as the N^{th} **partial sum of the Fourier series** of $[f]$. It follows from Theorem 74 that

$$[f] = \lim_{N \to \infty} \Delta_N([f]),$$

where the convergence is relative to the $\|\cdot\|_2$-norm which was mentioned above.

Note 16.2.1

This is a beautiful occurrence, having functions that can be written uniquely (up to a set of measure zero) as a linear combination of the ONB $\{[\xi_n]\}_{n \in \mathbb{Z}}$, which is a very powerful result that is often used in linear algebra.

We can then ask the question of whether the same result holds for other similarly defined $L_p(\mathbb{T}, \mathbb{C})$, for $1 \leq p \leq \infty$ where $p \neq 2$. We shall focus on L_1. Unfortunately, we shall see that this doesn't hold. The rest of the course is dedicated to showing this.

Notation

We shall note down here notations and definitions of which we've seen but require some modification for the purposes of our discussion.

- $\mathrm{Trig}(\mathbb{T}, \mathbb{C}) := \mathrm{span}\{\xi_n : n \in \mathbb{Z}\} = \{\sum_{n=-N}^{N} \alpha_n \xi_n : \alpha_n \in \mathbb{C} : N \in \mathbb{N} \setminus \{0\}$;
- $\mathcal{C}(\mathbb{T}, \mathbb{C}) := \{f : \mathbb{R} \to \mathbb{C} : f \text{ is continuous and } 2\pi\text{-periodic}\}$;
- $\mathrm{SIMP}(\mathbb{T}, \mathbb{C}) := \{f : \mathbb{R} \to \mathbb{C} : f \restriction_{[-\pi,\pi)} \text{ is a simple function and } f \text{ is } 2\pi\text{-periodic}\}$;
- $\mathrm{STEP}(\mathbb{T}, \mathbb{C}) := \{f : \mathbb{R} \to \mathbb{C} : f \restriction_{[-\pi,\pi)} \text{ is a step function and } f \text{ is } 2\pi\text{-periodic}\}$;
- *for* $1 \leq p < \infty$,

$$\mathcal{L}_p(\mathbb{T}, \mathbb{C}) := \{f : \mathbb{R} \to \mathbb{C} : f \text{ is measurable, } 2\pi\text{-periodic, and } \int_{[-\pi,\pi)} |f|^p < \infty\};$$

- *and for* $p = \infty$,

$$\mathcal{L}_p(\mathbb{T}, \mathbb{C}) := \{ f : \mathbb{R} \to \mathbb{C} : f \textit{ is measurable, } 2\pi\textit{-periodic, and essentially bounded } \};$$

Note that

$$\mathrm{Trig}(\mathbb{T}, \mathbb{C}) \subseteq \mathcal{C}(\mathbb{T}, \mathbb{C}) \subseteq \mathcal{L}_p(\mathbb{T}, \mathbb{C}), \quad 1 \leq p \leq \infty.$$

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17.1 Introduction to Fourier Analysis (Continued)

As was the case with $p = 2$, for each $1 \leq p < \infty$, $\mathcal{L}_p(\mathbb{T}, \mathbb{C})$ forms a vector space over $\mathbb{C}$, and the map

$$\nu_p : \mathcal{L}_p(\mathbb{T}, \mathbb{C}) \to \mathbb{R}$$
$$f \mapsto \left(\frac{1}{2\pi} \int_{[-\pi,\pi)} |f|^p \right)^{1/p}$$

defines a seminorm on $\mathcal{L}_p(\mathbb{T}, \mathbb{C})$

For $p = \infty$, echoing a similar argument as in Section 12.1.1, we have that

$$\nu_\infty(f) := \inf\{\delta > 0 : m\{\theta \in [-\pi, \pi) : |f(\theta)| > \delta\} = 0\},$$

for $f \in \mathcal{L}_\infty(\mathbb{T}, \mathbb{C})$ is a seminorm on $\mathcal{L}_\infty(\mathbb{T}, \mathbb{C})$.

By Proposition 44, for each $1 \leq p \leq \infty$, we can obtain a norm $\|\cdot\|_p$ on

$$L_p(\mathbb{T}, \mathbb{C}) := \mathcal{L}_p(\mathbb{T}, \mathbb{C}) \big/ \mathcal{N}_p(\mathbb{T}, \mathbb{C}),$$

where

$$\mathcal{N}_p(\mathbb{T}, \mathbb{C}) := \{f \in \mathcal{L}_p(\mathbb{T}, \mathbb{C}) : \nu_p(f) = 0\}.$$

[1] Again, we can find that $[f] = [g] \in L_p(\mathbb{T}, \mathbb{C})$ iff $f = g$ a.e. on $\mathbb{R}$.

[1] claftosel

Exercise 17.1.1

Verify that for $f \in \mathcal{C}(\mathbb{T}, \mathbb{C})$

$$\|[f]\|_\infty = \|f\|_{\sup} := \sup\{|f(\theta)| : \theta \in [-\pi, \pi)\}.$$

Note 17.1.1

Note that the supremum on the RHS of the above equation is a finite number, since $f \in \mathcal{C}(\mathbb{T}, \mathbb{C})$ implies that f is continuous on $\mathbb{R}$, and hence f is bounded on $[-\pi, \pi] \supseteq [-\pi, \pi)$.

Given any function $f : [-\pi, \pi) \to \mathbb{C}$,, let $\check{f} : \mathbb{R} \to \mathbb{C}$ be the 2π-periodic extension of f; i.e. $\check{f}(\theta) = f(\theta)$ for $\theta \in [-\pi, \pi)$ and $\check{f}(\theta + 2\pi) = \check{f}(\theta)$ for $\theta \in \mathbb{R}$. It is clear that $\check{f}$ always exists and is uniquely defined by f .

Theorem 76 (The 2π periodic extension map is an isometric isomorphism)

Let $1 \leq p \leq \infty$. The map

$$\begin{aligned} \Xi_p : L_p([-\pi, \pi), \mathbb{C}) &\to L_p(\mathbb{T}, \mathbb{C}) \\ [f] &\mapsto [\check{f}] \end{aligned}$$

is an isometric isomorphism.

Exercise 17.1.2

Prove Theorem 76.

It follows from the above isometric isomorphism that all of our results about L_p-spaces hold for their respective $L_p(\mathbb{T}, \mathbb{C})$ counterparts.

Let us now focus on $L_1(\mathbb{T}, \mathbb{C})$.

Definition 49 (The Fourier Coefficients and The Fourier Series)

For $f \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$ and $n \in \mathbb{Z}$, we refer to

$$\hat{f}(n) := \frac{1}{2\pi} \int_{[-\pi,\pi)} f \overline{\xi_n}$$

*as the n^{th}-**Fourier coefficient** of f. We also refer to*

$$\sum_{n \in \mathbb{Z}} \hat{f}(n) \xi_n$$

*as the **Fourier series** of f in $\mathcal{L}1(\mathbb{T}, \mathbb{C})$.*

Remark 17.1.1

If $f, g \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$ and $f = g$ a.e. on $[-\pi, \pi)$, then

$$\hat{f}(n) = \frac{1}{2\pi} \int_{[-\pi,\pi)} f\overline{\xi_n} = \frac{1}{2\pi} \int_{[-\pi,\pi)} g\overline{\xi_n} = \hat{g}(n), \quad \forall n \in \mathbb{Z}.$$

*Thus, if we set the n^{th}-**Fourier coefficient** of $[f] \in L_1(\mathbb{T}, \mathbb{C})$ as*

$$\alpha_n^{[f]} := \hat{f}(n), \quad n \in \mathbb{Z},$$

as we did in Example 16.2.1, then $\alpha_n^{[f]}$ is well-defined. We can thus define

$$\sum_{n \in \mathbb{Z}} \alpha_n^{[f]} [\xi_n]$$

*as the **Fourier series** of $[f]$.*

Notice that we did not mention the convergence of the above series. Up to now, the Fourier series is simply a formal power series, meant only to represent the sequence of partial sums

$$\left(\sum_{n=-N}^{N} \alpha_n^{[f]} [\xi_n] \right)_{N=0}^{\infty}.$$

We shall study about the convergence of the series.

Note that we may extend the notion of a Fourier coefficient for non-integer powers of $e^{i\theta}$; i.e. for $f \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$ and $r \in \mathbb{R}$, we define

$$\hat{f}(r) = \frac{1}{2\pi} \int_{[-\pi,\pi)} f\overline{\xi_r},$$

where $\xi_r(\theta) = e^{ir\theta}$ for all $\theta \in \mathbb{R}$.

Remark 17.1.2

In the case of $p = 2$, we've seen that $\left(\alpha_n^{[f]}\right)_{n \in \mathbb{Z}} \in \ell_2(\mathbb{Z}, \mathbb{C})$. While this does *not hold for $[f] \in L_1(\mathbb{T}, \mathbb{C})$, we can actually get pretty close.*

First, notice that $|\xi_r(\theta)| = 1$ for all $\theta \in \mathbb{R}$ and $r \in \mathbb{R}$. Thus for $f \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$,

$$\begin{aligned} \left|\hat{f}(n)\right| &= \left|\frac{1}{2\pi}\int_{[-\pi,\pi)} f\overline{\xi_r}\right| \\ &\leq \frac{1}{2\pi}\int_{[-\pi,\pi)} \left|f\overline{\xi_r}\right| \\ &= \frac{1}{2\pi}\int_{[-\pi,\pi)} |f| \\ &= \nu_1(f) = \|[f]\|_1 . \end{aligned}$$

So as before, if $f, g \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$, and $f = g$ a.e. on $[\mathbb{T}]$, then $\hat{f}(r) = \hat{g}(r)$ for all $r \in \mathbb{R}$. Thus, we may define $\alpha_r^{[f]} := \hat{f}(r)$, for $r \in \mathbb{R}$.

It follows that

$$\sup_{r\in\mathbb{R}} \left|\alpha_r^{[f]}\right| = \sup_{r\in\mathbb{R}} \left|\hat{f}(r)\right| \leq \|[f]\|_1$$

for all $[f] \in L_1(\mathbb{T}, \mathbb{C})$. In particular, we have that

$$\left(\alpha_n^{[f]}\right)_{n\in\mathbb{Z}} \in \ell_\infty(\mathbb{Z}, \mathbb{C}).$$

We can, in fact, do better.

Let

$$c_0(\mathbb{Z}, \mathbb{C}) := \left\{ (z_n)_{n=1}^\infty : \forall n \in \mathbb{N}\ z_n \in \mathbb{C} \wedge \lim_{n\to\infty} z_n = 0 \right\}.$$

Theorem 77 (The Riemann-Lebesgue Lemma)

Let $f \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$. Then

$$\lim_{r\to\infty} \hat{f}(r) = 0 = \lim_{r\to-\infty} \hat{f}(r).$$

In particular,

$$\left(\alpha_n^{[f]}\right)_{n\in\mathbb{Z}} \in c_0(\mathbb{Z}, \mathbb{C}).$$

Strategy

The key here is to realize that this is simple in the case of characteristic functions of an interval. Since the Lebesgue integration is linear, the span of the 2π-periodic extensions of these characteristic functions of intervals is $\mathrm{STEP}(\mathbb{T}, \mathbb{C})$. *The result would hold for its equivalence classes, and we then simply need to appeal to the density of* $[\mathrm{STEP}(\mathbb{T}, \mathbb{C})]$ *in* $L_1(\mathbb{T}, \mathbb{C})$, *which then gives us the result.*

Proof

Case: Characteristic functions Let f_0 be the characteristic of an interval $[s, t] \subseteq [-\pi, \pi)$, i.e. $f_0 = \chi_{[s,t]}$. Let $f := \check{f}_0$ be the 2π-periodic extension of f_0 to $\mathbb{R}$, so that $f \in \mathrm{STEP}(\mathbb{T}, \mathbb{C})$. Then f is continuous and, in particular, bounded, over a bounded interval, f is Riemann integrable as well. Then

$$\begin{aligned}\hat{f}(r) &= \frac{1}{2\pi} \int_{[-\pi,\pi)} \chi_{[s,t]} \overline{\xi_r} \\ &= \frac{1}{2\pi} \int_s^t e^{-ir\theta} \, d\theta \\ &= \frac{1}{2\pi} \left(\frac{e^{-irt} - e^{-irs}}{-ir} \right).\end{aligned}$$

Thus

$$\left| \hat{f}(r) \right| \leq \frac{\left| e^{-irt} \right| + \left| e^{-irs} \right|}{2\pi \left| -ir \right|} = \frac{2}{2\pi \left| r \right|} = \frac{1}{\pi \left| r \right|}.$$

It is clear that

$$\lim_{r \to \infty} \hat{f}(r) = 0 = \lim_{r \to -\infty} \hat{f}(r).$$

Case: Step functions Let $f \in \mathrm{STEP}(\mathbb{T}, \mathbb{C})$, and $f_0 := f \restriction_{[-\pi,\pi)}$, and write $f_0 = \sum_{k=1}^{M} \beta_k \chi_{H_k}$ as a disjoint representation, where each $H_k = [s_k, t_k]$ is a subinterval $[-\pi, \pi)$.

Then the result follows almost exactly like the last case for characteristic functions, while making use of the linearity of the Lebesgue integral.

Case: Final, generic case Let $[f] \in L_1(\mathbb{T}, \mathbb{C})$ and $\varepsilon > 0$. By the density of $[\mathrm{STEP}(\mathbb{T}, \mathbb{C})]$ in $L_1(\mathbb{T}, \mathbb{C})$, let $g \in \mathrm{STEP}(\mathbb{T}, \mathbb{C})$ such that

$$\left\| [f] - [g] \right\|_1 \leq \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned}\hat{f}(r) &= \frac{1}{2\pi}\int_{[-\pi,\pi)} f\overline{\xi_r} \\ &= \frac{1}{2\pi}\int_{[-\pi,\pi)} (f-g)\overline{\xi_r} + \frac{1}{2\pi}\int_{[-\pi,\pi)} g\overline{\xi_r} \\ &= \widehat{f-g}(r) + \hat{g}(r).\end{aligned}$$

As seen before, we have that

$$\left|\widehat{f-g}(r)\right| \leq \nu_1(f-g) = \|[f-g]\|_1 = \|[f]-[g]\|_1 < \frac{\varepsilon}{2}.$$

Now from the previous case, since $g \in \mathrm{STEP}(\mathbb{T},\mathbb{C})$, we may choose $N > 0$ such that $|r| > N$ so that $|\hat{g}(r)| < \frac{\varepsilon}{2}$. Thus $|r| > N$ implies that

$$\left|\hat{f}\right| \leq \left|\widehat{f-g}(r)\right| + |\hat{g}(r)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus

$$\lim_{r\to\infty} \hat{f}(r) = 0 = \lim_{r\to-\infty} \hat{f}(r),$$

as required.

Recall that

$$\alpha_n^{[f]} = \hat{f}(n), \quad n \in \mathbb{Z}.$$

It is clear that $\hat{f}(n) = \frac{1}{2\pi}\int_{[-\pi,\pi)} f\overline{\xi_n} \in \mathbb{C}$ and so $\alpha_n^{[f]} \in \mathbb{C}$, and

$$\lim_{n\to\infty} \alpha_n^{[f]} = \lim_{n\to\infty} \hat{f}(n) = 0.$$

Thus $\left(\alpha_n^{[f]}\right)_{n\in\mathbb{Z}} \in c_0(\mathbb{Z},\mathbb{C})$. □

Remark 17.1.3

Recall that we had $[f] \in L_2(\mathbb{T},\mathbb{C})$ iff $(\alpha_n^{[f]})_{n\in\mathbb{Z}} \in \ell_2(\mathbb{Z},\mathbb{C})$. We have shown that if $[f] \in L_1(\mathbb{T},\mathbb{C})$ implies that

$$(\alpha_n^{[f]})_{n\in\mathbb{Z}} \in c_0(\mathbb{Z},\mathbb{C}).$$

However, the converse is not true. We shall see in the final chapter that the map

$$\Lambda : (L_1(\mathbb{T},\mathbb{C}), \|\cdot\|_1) \to (c_0(\mathbb{Z},\mathbb{C}), \|\cdot\|_\infty)$$

$$[f] \mapsto \left(\alpha_n^{[f]}\right)_{n\in\mathbb{Z}}$$

is a continuous, injective linear map, but it is ***not surjective****.*

We are left with some other questions as well; for $[f] \in L_1(\mathbb{T}, \mathbb{C})$:

1. does the Fourier series $\sum_{n\in\mathbb{Z}} \alpha_n^{[f]}[\xi_n]$ of f converge, and if so, in which sense? Is it pointwise (a.e.), uniformly, or in the L_1-norm?
2. if the Fourier series does converge in some sense, is the value f itself?
3. Is $[f]$ completely determined by its Fourier series, as we have seen for L_2? That is, if $[f], [g] \in L_1(\mathbb{T}, \mathbb{C})$, and $\alpha_n^{[f]} = \alpha_n^{[g]}$ for all $n \in \mathbb{Z}$, is it true that $[f] = [g]$?

17.2 Convolution

Recall that an algebra is a vector space over some field $\mathbb{F}$ which also happens to be a ring. A **Banach algebra** $\mathcal{A}$ is a Banach space over $\mathbb{K}$ which is also an algebra, where multiplication is jointly continuous since it satisfies the inequality

$$\|ab\| \leq \|a\| \, \|b\|$$

for all $a, b \in \mathcal{A}$.

Example 17.2.1

$(\mathcal{C}(X, \mathbb{K}), \|\cdot\|_{\sup})$ is a Banach algebra for each locally compact, Hausdorff topological space X. In particular, $\mathbb{M}_n(\mathbb{K}) \simeq \mathcal{B}(\mathbb{K}^n)$ (for each $n \geq 1$), when equipped with the operator norm, is a Banach algebra.

Thus far, we have seen that $L_1(\mathbb{T}, \mathbb{C})$ is a Banach space, but we have not studied about whether it has any multiplicative structure.

For $f, g \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$, we set

$$g \diamond f(\theta) := \frac{1}{2\pi} \int_{[-\pi,\pi)} g(s) f(\theta - s) \, \mathrm{dm}(s),$$

where $\mathrm{dm}(s)$ is similar to the notation dx in Riemann integration, only as an indicator of the variable of which the integration is performed with respect to (wrt). We refer to $g \diamond f(\theta)$ as the **convolution** of g and f. Observe that if $[f_1] = [f_2], [g_1] = [g_2] \in L_1(\mathbb{T}, \mathbb{C})$, then $g_1 \diamond f_1 = g_2 \diamond f_2$ a.e., and so we may define

$$[g] * [f] := [g \diamond f],$$

for all $[f], [g] \in L_1(\mathbb{T}, \mathbb{C})$.

The careful reader would quickly notice the following 2 points:

1. it is not clear that $g \diamond f \in \mathbb{C}$ for any $\theta \in \mathbb{R}$;
2. it is much less clear that $g \diamond f \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$.

To prove the above statement, we require **Fubini's Theorem**, which requires quite a bit of work.

We shall work around Fubini's Theorem due to the overhead that we have to take on. Instead, we shall instead show that we can turn $\mathcal{L}_1(\mathbb{T}, \mathbb{C})$ (and in turn $L_1(\mathbb{T}, \mathbb{C})$) into a so-called **left module** over $\mathcal{C}(\mathbb{T}, \mathbb{C})$ using convolution. [2] That is, given $g \in \mathcal{C}(\mathbb{T}, \mathbb{C})$ and $f \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$, we shall set

$$g \diamond f(\theta) := \frac{1}{2\pi} \int_{[-\pi,\pi)} g(s) f(\theta - s) \, \mathrm{dm}(s),$$

and prove that $g \diamond f \in \mathcal{C}(\mathbb{T}, \mathbb{C}) \subseteq \mathcal{L}_1(\mathbb{T}, \mathbb{C})$. [3] Now if this is true, then if $f_1 \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$ and $f_1 = f$ a.e. on $\mathbb{R}$, then $g \diamond f(\theta) = g \diamond f_1(\theta)$ for all $\theta \in \mathbb{R}$, then $g \diamond f = g \diamond f_1$, and we can thus define

$$g * [f] = [g \diamond f], \quad [f] \in L_1(\mathbb{T}, \mathbb{C}).$$

[2] Wikipedia article for left module.

[3] Note $\mathcal{C}(\mathbb{T}, \mathbb{C}) \subseteq \mathcal{L}_p(\mathbb{T}, \mathbb{C})$ for $1 \leq p \leq \infty$.

One advantage to convolving with continuous functions only is that we can make use of the Riemann integral. This will allow us to garner more information about the continuity properties of $\diamond$, and ultimately about convergence properties of the Fourier series.

Lemma 78 (Preservation of the Lebesgue Integral of 2π-periodic functions under certain Transformations)

Let $f \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$ and $\theta \in \mathbb{R}$.

1. $$\int_{[-\pi,\pi)} f = \int_{[-\pi,\pi)} \tau_s^\circ(f),$$
   where $\tau_s^\circ(f)(\theta) = f(\theta - s)$ *is a translation of* f *by* s.

2. *If* $h(s) := f(-s)$, $s \in \mathbb{R}$, *is a reflection of* f *(on some axis), then*
   $$\int_{[-\pi,\pi)} h = \int_{[-\pi,\pi]} f.$$

3. *Let* $\varphi_{f;\theta} : \mathbb{R} \to \mathbb{C}$ *be* $\varphi_{f,\theta}(s) = f(\theta - s)$. *Then* $\varphi_{f,\theta} \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$ *and*
   $$\nu_1(\varphi_{f,\theta}) = \nu_1(f).$$
   That is,
   $$\frac{1}{2\pi} \int_{[-\pi,\pi)} |f(\theta - s)| \, \mathrm{dm}(s) = \frac{1}{2\pi} \int_{[-\pi,\pi)} |f(t)| \, \mathrm{dm}(t).$$

Proof

The proof of this lemma is in A6Q1. □

Definition 50 (Convolution)

Let $f \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$ *and* $g \in \mathcal{C}(\mathbb{T}, \mathbb{C})$. *We define the* ***convolution*** *of* f *by* g *to be the function*

$$\begin{aligned} g \diamond f &: \mathbb{R} \to \mathbb{C} \\ \theta &\mapsto \frac{1}{2\pi} \int_{[-\pi,\pi)} g(s) f(\theta - s) \, \mathrm{dm}(s). \end{aligned}$$

We still have not shown that $g \diamond f(\theta) \in \mathbb{C}$ for each $\theta \in \mathbb{R}$. Let's do that right now.

Fixing $\theta \in \mathbb{R}$, we see that by Lemma 78,

$$|g \diamond f(\theta)| = \left| \frac{1}{2\pi} \int_{[-\pi,\pi)} g(s) f(\theta - s) \, \mathrm{dm}(s) \right|$$

$$\begin{aligned}
&\leq \frac{1}{2\pi} \int_{[-\pi,\pi)} |g(s)| \, |f(\theta - s)| \, \mathrm{d}m(s) \\
&\leq \|g\|_{\sup} \frac{1}{2\pi} \int_{[-\pi,\pi)} \left| \varphi_{f,\theta}(s) \right| \mathrm{d}m(s) \\
&= \|g\|_{\sup} \, \nu_1(\varphi_{f,\theta}) \\
&= \|g\|_{\sup} \, \nu_1(f) < \infty.
\end{aligned}$$

It follows that $g \diamond f$ is indeed a complex-valued function.

The following is an extremely important lemma that we shall use extensively.

Lemma 79 (Swapping Convolutions)

Let $f \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$ and $g \in \mathcal{L}_\infty(\mathbb{T}, \mathbb{C})$. If $\theta \in \mathbb{R}$, then

$$\int_{[-\pi,\pi)} g(s) f(\theta - s) \, \mathrm{d}m(s) = \int_{[-\pi,\pi)} g(\theta - t) f(t) \, \mathrm{d}m(t).$$

In particular, this holds if

$$f \in \mathcal{L}_1(\mathbb{T}, \mathbb{C}) \quad \text{and} \quad g \in \mathcal{C}(\mathbb{T}, \mathbb{C}).$$

Proof

The proof of this lemma is in A6Q2. □

Remark 17.2.1

With Lemma 79, for $f \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$ and $g \in \mathcal{C}(\mathbb{T}, \mathbb{C})$, we can define the ***convolution*** *of g by f as*

$$f \diamond g(\theta) = \frac{1}{2\pi} \int_{[-\pi,\pi)} g(\theta - t) f(t) \, \mathrm{d}m(t).$$

Consequently, we have that $f \diamond g(\theta) = g \diamond f(\theta)$ for all $\theta \in \mathbb{R}$, and so we shall simply refer to this function as the convolution of f and g.

Exercise 17.2.1

Let $h : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic and continuous function. Prove that h is uniformly continuous. [4]

[4] This is a rather simple (even proof-wise) but important realization in our theories going forward.

Proposition 80 (Continuity of the Convolution of f and g where g is Continuous)

Let $g \in \mathcal{C}(\mathbb{T}, \mathbb{C})$ and $f \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$. Then $g \diamond f \in \mathcal{C}(\mathbb{T}, \mathbb{C})$.

Proof

First, note that by Exercise 17.2.1, g is uniformly continuous. Let $\varepsilon > 0$. We can then choose $\delta > 0$ such that $\forall x, y \in \mathbb{R}$, $|x - y| < \delta$ implies that $|g(x) - g(y)| < \frac{\varepsilon}{\nu_1(f)}$.

Now for any $\theta, \theta_0 \in \mathbb{R}$ such that $|\theta - s - (\theta_0 - s)| = |\theta - \theta_0| < \delta$, for any $s \in \mathbb{R}$, we have that

$$|g(\theta - s) - g(\theta_0 - s)| < \frac{\varepsilon}{\nu_1(f)}.$$

Then by Lemma 79 and the last remark, we have

$$\begin{aligned}
&|g \diamond f(\theta) - g \diamond f(\theta_0)| \\
&= \frac{1}{2\pi} \left| \int_{[\pi,\pi)} g(\theta - s) f(s) - g(\theta_0 - s) f(s) \, \mathrm{dm}(s) \right| \\
&\leq \frac{1}{2\pi} \int_{[-\pi,\pi)} |g(\theta - s) - g(\theta_0 - s)| \, |f(s)| \, \mathrm{dm}(s) \\
&< \frac{1}{2\pi} \int_{[-\pi,\pi)} \frac{\varepsilon}{\nu_1(f)} \, |f(s)| \, \mathrm{dm}(s) \\
&= \frac{\varepsilon}{\nu_1(f)} \nu_1(f) = \varepsilon.
\end{aligned}$$

Thus $g \diamond f$ is (uniformly) continuous.

That $g \diamond f$ is 2π-periodic follows from g and f being 2π-periodic themselves. □

We now want to see if given $[f_1] = [f_2] \in L_1(\mathbb{T}, \mathbb{C})$, do we have $g \diamond f_1 = g \diamond f_2$? This is, in particular, motivated by what we already saw in $L_2(\mathbb{T}, \mathbb{C})$, where this realization allowed us to work solely with $L_2(\mathbb{T}, \mathbb{C})$ instead of $\mathcal{L}_2(\mathbb{T}, \mathbb{C})$. Fortunately, this indeed holds for $L_1(\mathbb{T}, \mathbb{C})$.

Observe that if $[f_1] = [f_2] \in L_1(\mathbb{T}, \mathbb{C})$, then $f_1 = f_2$ a.e. on $\mathbb{R}$, which then $gf_1 = gf_2$ a.e. We thus see that $\forall \theta \in \mathbb{R}$, and any $s \in \mathbb{R}$,

$$\begin{aligned} g \diamond f_1(\theta) &= f_1 \diamond g(\theta) \\ &= \frac{1}{2\pi} \int_{[-\pi,\pi)} g(\theta - s) f_1(s) \, \mathrm{d}m(s) \\ &= \frac{1}{2\pi} \int_{[-\pi,\pi]} g(\theta - s) f_2(s) \, \mathrm{d}m(s) \\ &= f_2 \diamond g(\theta) = g \diamond f_2(\theta). \end{aligned}$$

We may thus extend our notion of convolutions to $L_1(\mathbb{T}, \mathbb{C})$.

Definition 51 (Convolution on $L_1(\mathbb{T}, \mathbb{C})$)

*Given $g \in \mathcal{C}(\mathbb{T}, \mathbb{C})$ and $[f] \in L_1(bt, \mathbb{C})$, we define the **convolution** of g and $[f]$ to be*

$$g * [f] := [g \diamond f],$$

where $g \diamond f \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$ is the convolution introduced in Definition 50.

Definition 52 (Convolution Operator with Kernel)

*We define the **convolution operator** with **kernel** g to be the map*

$$\begin{aligned} C_g : L_1(\mathbb{T}, \mathbb{C}) &\to L_1(\mathbb{T}, \mathbb{C}) \\ [f] &\mapsto g * [f]. \end{aligned}$$

Warning

The kernel defined above has nothing to do with the notion of kernels in abstract algebra.

Remark 17.2.2

Observe that if $[f_1], f_2 \in L_1(\mathbb{T}, \mathbb{C})$, and if $\kappa \in \mathbb{C}$, then

$$\begin{aligned} C_g(\kappa[f_1] + [f_2]) &= g * [\kappa f_1 + f_2] \\ &= \frac{1}{2\pi} \int_{[-\pi,\pi)} g(s)(\kappa f_1(\theta - s) + f_2(\theta - s)) \,\mathrm{dm}(s) \\ &= \kappa \frac{1}{2\pi} \int_{[-\pi,\pi)} g(s) f_1(\theta - s) \,\mathrm{dm}(s) \\ &\quad + \frac{1}{2\pi} \int_{[-\pi,\pi)} g(s) f_2(\theta - s) \,\mathrm{dm}(s) \\ &= \kappa g * [f_1] + g * [f_2] \\ &= \kappa C_g([f_1]) + C_g([f_2]). \end{aligned}$$

Thus C_g is a linear map on $L_1(\mathbb{T}, \mathbb{C})$.

Since $(L_1(\mathbb{T}, \mathbb{C}), \|\cdot\|_1)$ is a Banach space, and C_g is linear, it is natural to ask if C_g is bounded, [5] *and if so, what is its operator norm?*

We shall see that the answer to this question is intimately related to the question of convergence of Fourier series of elements of $L_1(\mathbb{T}, \mathbb{C})$.

[5] This would also mean that C_g is continuous.

With our current tool set, it is rather difficult to directly compute $\|C_g\|$. In particular, we have to deal with monstrosities of the following form:

$$\frac{1}{2\pi} \int_{[-\pi,\pi)} \left(\frac{1}{2\pi} \int_{[-\pi,\pi)} g(s) f(\theta - s) \,\mathrm{dm}(s) \right) e^{-in\theta} \,\mathrm{dm}(\theta).$$

What we shall do is to reformulate C_g as a vector-valued Riemann integral on $L_1(\mathbb{T}, \mathbb{C})$. We shall be able to extend this notion of convolution beyond the Banach space $L_1(\mathbb{T}, \mathbb{C})$. To that end, we first need to understand the notion of a **homogeneous Banach space**.

18 Lecture 18 Jul 11th 2019

18.1 Convolution (Continued)

Let $f \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$, and $s \in \mathbb{R}$. Consider the function

$$\begin{aligned} \tau_s^\circ(f) : \mathbb{R} &\to \mathbb{C} \\ \theta &\mapsto f(\theta - s), \end{aligned}$$

which we have seen before. One should think of τ_s° as translating f by s. The superscript $\circ$ above τ_s is to indicate that we are acting on functions. When working with elements of $L_1(\mathbb{T}, \mathbb{C})$, we shall drop this superscript.

Now, since $\mathfrak{M}(\mathbb{R})$ is invariant under translation, the Lebesgue measure is translation-invariant, and the set of 2π-periodic functions is invariant under translation implies that

$$\tau_s^\circ(f) \in \mathcal{L}_1(\mathbb{T}, \mathbb{C}).$$

Furthermore, if $[f] = [g] \in L_1(\mathbb{T}, \mathbb{C})$, then

$$[\tau_s^\circ(f)] = [\tau_s^\circ(g)].$$

Thus, we may define the operation of **translation by s** on $L_1(\mathbb{T}, \mathbb{C})$ as

$$\tau_s([f]) := [\tau_s^\circ(f)].$$

Definition 53 (Homogeneous Banach Spaces)

*A **homogeneous Banach space** over $\mathbb{T}$ is a linear manifold $\mathfrak{B}$ in $L_1(\mathbb{T}, \mathbb{C})$,* *equipped with the norm $\|\cdot\|_{\mathfrak{B}}$ wrt to which $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$ is a Banach space, satisfying*

1. $\|[f]\|_1 \leq \|[f]\|_{\mathfrak{B}}$ *for all* $[f] \in \mathfrak{B}$;
2. $[\mathrm{Trig}(\mathbb{T}, \mathbb{C})] \subseteq \mathfrak{B}$;
3. $\mathfrak{B}$ *is invariant under translation; i.e.* $\forall [f] \in \mathfrak{B}$ *and* $s \in \mathbb{R}$,
$$\tau_s[f] = [\tau_s^\circ(f)] \in \mathfrak{B};$$
4. $\forall [f] \in \mathfrak{B}$, $s \in \mathbb{R}$, $\|\tau_s[f]\|_{\mathfrak{B}} = \|[f]\|_{\mathfrak{B}}$; *and*
5. *for each* $[f] \in \mathfrak{B}$, *the map*
$$\begin{aligned} \Psi_{[f]} : \mathbb{R} &\to \mathfrak{B} \\ s &\mapsto \tau_s[f] \end{aligned}$$
is continuous. [1]

[1] This means that the translation itself is a continuous process on a homogeneous Banach space.

Remark 18.1.1

It may be surprising to find that a linear manifold $\mathfrak{M}$ of a Banach space $\mathfrak{X}$ may not be closed in the ambient norm, but that $(\mathfrak{M}, \|\cdot\|_{\mathfrak{M}})$ is complete in its own norm.

But one may quickly notice that each of the spaces $L_p(\mathbb{T}, \mathbb{C})$ is dense in $L_1(\mathbb{T}, \mathbb{C})$, for $1 \leq p < \infty$, and each of them is complete under their corresponding $\|\cdot\|_p$-norm. So we've already seen the above 'surprising' fact.

Example 18.1.1 (($[\mathcal{C}(\mathbb{T}, \mathbb{C})], \|\cdot\|_\infty$) is a homogeneous Banach space)

Recall that
$$[\mathcal{C}(\mathbb{T}, \mathbb{C})] \subseteq L_\infty(\mathbb{T}, \mathbb{C})$$
is a subset of $L_1(\mathbb{T}, \mathbb{C})$ and it is a linear manifold. Furthermore, for $f \in \mathcal{C}(\mathbb{T}, \mathbb{C})$, we have that
$$\|[f]\|_\infty = \|f\|_{\sup} := \sup\{|f(\theta)| : \theta \in [-\pi, \pi)\},$$
and that $([\mathcal{C}(\mathbb{T}, \mathbb{C})], \|\cdot\|_\infty)$ is a Banach space. We shall show that it is, in fact, a homogeneous Banach space.

1. Let $[f] \in [\mathcal{C}(\mathbb{T}, \mathbb{C})]$. Then

$$\|[f]\|_1 = \frac{1}{2\pi} \int_{[-\pi,\pi)} |f| \leq \frac{1}{2\pi} \int_{[-\pi,\pi)} \|f\|_{\sup} = \|f\|_{\sup} = \|[f]\|_\infty .$$

2. It is clear that $\xi_n(\theta) = e^{in\theta}$ is continuous for each $\theta \in \mathbb{R}$ and so $\xi_n \in \mathcal{C}(\mathbb{T}, \mathbb{C})$. Since $[\mathcal{C}(\mathbb{T}, \mathbb{C})]$ is a linear manifold, it follows that $[\mathrm{Trig}(\mathbb{T}, \mathbb{C})] \subseteq [\mathcal{C}(\mathbb{T}, \mathbb{C})]$.

3. If $f \in \mathcal{C}(\mathbb{T}, \mathbb{C})$, then it is clear that $\tau_s^\circ(f) \in \mathcal{C}(\mathbb{T}, \mathbb{C})$, since a translation of 2π-periodic continuous function is still a 2π-periodic continuous function. Thus $[\mathcal{C}(\mathbb{T}, \mathbb{C})]$ is translation invariant.

4. Let $[f] \in [\mathcal{C}(\mathbb{T}, \mathbb{C})]$. Then

$$\begin{aligned}
\|\tau_s[f]\|_\infty &= \|[\tau_s^\circ(f)]\|_\infty = \|\tau_s^\circ(f)\|_\infty \\
&= \sup\{|f(\theta - s)| : \theta \in \mathbb{R}\} \\
&= \sup\{|f(\theta)| : \theta \in \mathbb{R}\} \\
&= \|f\|_{\sup} = \|[f]\|_\infty .
\end{aligned}$$

5. Let $[f] \in [\mathcal{C}(\mathbb{T}, \mathbb{C})]$, and wlog wma $f \in \mathcal{C}(\mathbb{T}, \mathbb{C})$. Since f is continuous, we have that for every $s \in \mathbb{R}$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall s_0 \in \mathbb{R}$, if $|s - s_0| < \delta$, then $|f(s) - f(s_0)| < \varepsilon$. In particular, for any $\theta \in \mathbb{R}$, since $|\theta - s - (\theta - s_0)| < \delta$, we have

$$|f(\theta - s) - f(\theta - s_0)| < \frac{\varepsilon}{2}.$$

   Now for any $s \in \mathbb{R}$, and any $\varepsilon > 0$, we may pick the same $\delta > 0$ so that for any $s_0 \in \mathbb{R}$, we have

$$\begin{aligned}
\left\|\Psi_{[f]}(s) - \Psi_{[f]}(s_0)\right\|_\infty &= \|\tau_s[f] - \tau_{s_0}[f]\|_\infty \\
&= \left\|[\tau_s^\circ(f)] - [\tau_{s_0}^\circ(f)]\right\|_\infty \\
&= \left\|\tau_s^\circ(f) - \tau_{s_0}^\circ(f)\right\|_{\sup} \\
&= \|f(\theta - s) - f(\theta - s_0)\| \\
&= \sup\{|f(\theta - s) - f(\theta - s_0)| : \theta \in \mathbb{R}\} \\
&\leq \sup\left\{\frac{\varepsilon}{2} : \theta \in \mathbb{R}\right\} = \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

   It follows that $\Psi_{[f]}$ is indeed continuous for every $s \in \mathbb{R}$.

This concludes the proof that $([\mathcal{C}(\mathbb{T}, \mathbb{C})], \|\cdot\|_\infty)$ is a homogeneous Banach space.

Example 18.1.2 ($(L_p(\mathbb{T}, \mathbb{C}), \|\cdot\|_p)$ is a homogeneous Banach space for $1 \leq p < \infty$)

Let $1 \leq p < \infty$. We shall show that $(L_p(\mathbb{T}, \mathbb{C}), \|\cdot\|_p)$ is a homogeneous Banach space.

1. Let $f \in \mathcal{L}_p(\mathbb{T}, \mathbb{C})$, and q the Lebesgue conjugate of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Recall from Proposition 23 that there exists a measurable function $\rho : \mathbb{R} \to \mathbb{T}$ such that $f = \rho\,|f|$. One may observe that by Hölder's Inequality, and the fact that f itself is 2π-periodic, we have $\rho \in \mathcal{L}_q(\mathbb{T}, \mathbb{C})$. Furthermore,

$$\|[\rho]\|_q = \left(\frac{1}{2\pi} \int_{[-\pi,\pi)} |\rho|^q \right)^{1/q} \leq \left(\frac{1}{2\pi} \int_{[-\pi,\pi)} 1 \right)^{1/q} = 1.$$

   Most importantly, for us here, $\|[\overline{\rho}]\|_q = \|[\rho]\|_q = 1$. It follows, again, by Holder's Inequality, that

$$\|[f]\|_p = \left(\frac{1}{2\pi} \int_{[-\pi,\pi)} |f \cdot \overline{\rho}| \right) \leq \|[f]\|_p \|[\overline{\rho}]\|_q \leq \|[f]\|_p .$$

2. As a consequence of the last example, we observe that

$$[\mathrm{Trig}(\mathbb{T}, \mathbb{C})] \subseteq [\mathcal{C}(\mathbb{T}, \mathbb{C})] \subseteq L_p(\mathbb{T}, \mathbb{C}) \subseteq L_1(\mathbb{T}, \mathbb{C}).$$

3. The fact that the norm is finite in next part makes the final conclusion.

4. Let $[f] \in L_p(\mathbb{T}, \mathbb{C})$ and $s \in \mathbb{R}$. We observe that by Lemma 79,

$$\begin{aligned} \|\tau_s[f]\|_p &= \|[\tau_s^\circ(f)]\|_p \\ &= \left(\frac{1}{2\pi} \int_{[-\pi,\pi)} |f(\theta - s)|^p \,\mathrm{dm}(s) \right)^{1/p} \\ &= \left(\frac{1}{2\pi} \int_{[-\pi,\pi)} |f(s)|^p \,\mathrm{dm}(s) \right)^{1/p} \\ &= \|[f]\|_p < \infty. \end{aligned}$$

5. Let $[f] \in L_p(\mathbb{T}, \mathbb{C})$ and $s \in \mathbb{R}$. WTS $\forall \varepsilon > 0, \exists \delta > 0$ such that

$\forall s_0 \in \mathbb{R}$, if $|s - s_0| < \delta$, then

$$\begin{aligned}\left\| \Psi_{[f]}(s) - \Psi_{[f]}(s_0) \right\|_p &= \|\tau_s[f] - \tau_{s_0}[f]\|_p \\ &= \left\| [\tau_s^\circ(f)] - [\tau_{s_0}^\circ(f)] \right\|_p \\ &= \left(\frac{1}{2\pi} \int_{[-\pi,\pi)} |f(\theta - s) - f(\theta - s_0)| \,\mathrm{d}m(s) \right)^{1/p}.\end{aligned}$$

We realize that we need to see if we can have $f(\theta - s)$ to be as close to $f(\theta - s_0)$ as possible under the right circumstances.

Notice that $[\mathcal{C}(\mathbb{T}, \mathbb{C})]$ is dense in $L_p(\mathbb{T}, \mathbb{C})$. Thus, we may find $[g] \in [\mathcal{C}(\mathbb{T}, \mathbb{C})]$ such that

$$\|[f] - [g]\|_p < \frac{\varepsilon}{3}.$$

Furthermore, we can pick this g such that $\exists \delta > 0$ such that for $|s - s_0| < \delta$, we have

$$\left\| \tau_s^\circ(g) - \tau_{s_0}^\circ(g) \right\|_\infty < \frac{\varepsilon}{3},$$

and this is by the last example. Note that

$$\|\cdot\|_1 \leq \|\cdot\|_p \leq \|\cot\|_\infty.$$

Thus by the same δ, we have

$$\begin{aligned}\left\| \tau_s^\circ(f) - \tau_{s_0}^\circ(f) \right\|_p &\leq \|\tau_s^\circ(f) - \tau_s^\circ(g)\|_p + \left\| \tau_s^\circ(g) - \tau_{s_0}^\circ(g) \right\|_p \\ &\quad + \left\| \tau_{s_0}^\circ(g) - \tau_{s_0}^\circ(f) \right\|_p \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

saving us the work of doing integration, and completing the proof.

Example 18.1.3 (**$(L_\infty(\mathbb{T}, \mathbb{C}), \|\cdot\|_\infty)$ is not a homogeneous Banach space**)

The situation for $p = \infty$ is different. It checks out the first 4 conditions, but fails on the last; translations under this norm is not continuous. That sounds sensible, given how the norm is defined as a supremum and not some nice elementary function, but we shall see where exactly does it fall short.

1. This is an easy exercise: for $[f] \in L_\infty(\mathbb{T}, \mathbb{C})$,

$$\|[f]\|_1 = \frac{1}{2\pi} \int_{[-\pi,\pi)} |f| \leq \frac{1}{2\pi} \int_{[-\pi,\pi)} \|f\|_{\sup} = \|f\|_{\sup} = \|[f]\|_\infty .$$

2. Again, by Example 18.1.1, we have that

$$[\mathrm{Trig}(\mathbb{T}, \mathbb{C})] \subseteq [\mathcal{C}(\mathbb{T}, \mathbb{C})] \subseteq L_\infty(\mathbb{T}, \mathbb{C}).$$

3. For $[f] \in L_\infty(\mathbb{T}, \mathbb{C})$ and $s \in \mathbb{R}$, we have

$$\begin{aligned} \|\tau_s[f]\|_\infty &= \|[\tau_s^\circ(f)]\|_\infty \\ &= \|f(\theta - s)\|_{\sup} \\ &= \sup\{|f(\theta - s)| : \theta \in \mathbb{R}\} \\ &= \sup\{|f(\theta)| : \theta \in \mathbb{R}\} \\ &= \|f\|_{\sup} = \|[f]\|_\infty < \infty. \end{aligned}$$

4. The last part concluded with what we want.

   For the translation not being continuous, consider the function

$$f_0 := \chi_{[0,\pi)} \in \mathcal{L}_\infty([-\pi, \pi), \mathbb{C})$$

and let

$$f = \check{f}_0 \in \mathcal{L}_\infty(\mathbb{T}, \mathbb{C})$$

be the 2π-periodic extension of f_0.

For $-\pi < s < 0$, we see that

$$\tau_s(f)(\theta) - \tau_0(f)(\theta) = 1 - 0 = 1$$

for all $\theta \in (s, 0)$, and so

$$\|\tau_s[f] - \tau_0[f]\|_\infty = 1.$$

In particular,

$$\lim_{s \to 0} \|\tau_s[f] - \tau_0[f]\|_\infty = 1 \neq 0 \, \|\tau_0[f] - \tau_0[f]\|_\infty ,$$

i.e. even if s is close to 0, the translation does not get any more continuous. Thus, in particular, $s \mapsto \tau_s[f]$ is not continuous at 0.

Let $g \in \mathcal{C}(\mathbb{T},\mathbb{C})$ and $[f] \in L_1(\mathbb{T},\mathbb{C})$. We defined the convolution of g and $[f]$ to be $g * [f] := [g \diamond f]$, where

$$g \diamond f(\theta) = \frac{1}{2\pi} \int_{[-\pi,\pi)} g(s) f(\theta - s) \,\mathrm{dm}(s).$$

So we defined $g * [f]$ by first defining $g \diamond f$ pointwise, using Lebesgue integration.

We showed in Example 18.1.2 that $(L_1(\mathbb{T},\mathbb{C}), \|\cdot\|_1)$ is a homogeneous Banach space over $\mathbb{T}$. Then by the 5^{th} condition in the definition, the function

$$\begin{aligned} \beta : \mathbb{R} &\to L_1(\mathbb{T},\mathbb{C}) \\ s &\mapsto g(s)\tau_s[f] \end{aligned}$$

is continuous. By Theorem 3,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \beta(s)\, ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s)\tau_s[f]\, ds$$

exists in $L_1(\mathbb{T},\mathbb{C})$, and it is obtained as an $\|\cdot\|_1$-limit of Riemann sums $(\beta, P_N, P_N^*) \in L_1(\mathbb{T},\mathbb{C})$ using partitions P_N of $[-2\pi, 2\pi]$ with corresponding choices P_N^* of test values for P_N.

Fixing $g \in \mathcal{C}(\mathbb{T},\mathbb{C})$, we can define the map

$$\begin{aligned} \Gamma_g : L_1(\mathbb{T},\mathbb{C}) &\to L_1(\mathbb{T},\mathbb{C}) \\ [f] &\mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s)\tau_s[f]\, ds. \end{aligned}$$

Notice that Γ_g is linear: for $[f], [h] \in L_1(\mathbb{T},\mathbb{C})$, we have

$$\begin{aligned} \Gamma_g([f] + [h]) &= \Gamma([f + h]) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s)\tau_s([f+h])\, ds \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s)[\tau_s^\circ(f+h)]\, ds \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s)[f(\theta - s) + h(\theta - s)]\, ds \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s)([f(\theta - s)] + [h(\theta - s)])\, ds \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s)\tau_s[f] + g(s)\tau_s[h] \, ds \\
&= \Gamma_g([f]) + \Gamma_g([h]).
\end{aligned}$$

One quickly realizes the resemblance of Γ_g to C_g. After all, in particular,

$$\tau_s[f] = [\tau_s^\circ(f)], \quad \text{and} \quad \tau_s^\circ(f)(\theta) = f(\theta - s),$$

for all $\theta \in \mathbb{R}$.

We shall make showing $\Gamma_g = C_g$ as our next goal, so that for $[f] \in L_1(\mathbb{T}, \mathbb{C})$, we have

$$C_g[f] = g * [f] = [g \diamond f] = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s)\tau_s[f] \, ds = \Gamma_g[f].$$

This is, however, not an obvious or trivial result, especially since the two constructions are entirely different; one is an equivalence class of convolutions, while the other is an integral of convolution-like expressions but involving equivalence classes.

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19.1 Convolution (Continued 2)

By Example 18.1.1, it follows that if $f \in \mathcal{C}(\mathbb{T}, \mathbb{C})$, then the map $s \mapsto \tau_s[f]$, or equivalently $s \mapsto \tau_s^\circ(f)$ is continuous from $(\mathbb{T}, |\cdot|)$ to $(\mathcal{C}(\mathbb{T}, \mathbb{C}), \|\cdot\|_{\sup})$.

Lemma 81 (Pointwise Value of Γ_g)

Let $f, g \in (\mathcal{C}(\mathbb{T}, \mathbb{C}), \|\cdot\|_{\sup})$. Let

$$\Gamma_g^\circ(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s) \tau_s^\circ(f) \, ds,$$

taken as a Banach space Riemann integral in $(\mathcal{C}(\mathbb{T}, \mathbb{C}), \|\cdot\|_{\sup})$. Then

$$\Gamma_g^\circ(f)(\theta) = g \diamond f(\theta) = \frac{1}{2\pi} \int_{[-\pi, \pi)} g(s) f(\theta - s) \, \mathrm{dm}(s)$$

for all $\theta \in \mathbb{R}$.

Strategy

The most difficult part of this proof is to understand the difference between $\Gamma_g^\circ(f)$ and $g \diamond f$. For $\Gamma_g^\circ(f)$, since $(\mathcal{C}(\mathbb{T}, \mathbb{C}), \|\cdot\|_{\sup})$ is a Banach space, and $\beta : \mathbb{R} \to \mathcal{C}(\mathbb{T}, \mathbb{C})$ given by $\beta(s) := g(s) f \tau_s^\circ(f) \in \mathcal{C}(\mathbb{T}, \mathbb{C})$ is continuous, by Theorem 3, $\Gamma_g^\circ(f)$ is a $\|\cdot\|_{\sup}$-limit of Riemann sums $S(\beta, P_N, P_n^*)$. We may further, wlog, suppose that for each $N \geq 1$, $P_N \in \mathcal{P}([-\pi, \pi])$ is a regular partition of $[-\pi, \pi]$ into 2^N subintervals of equal length $\frac{2\pi}{2^N}$, and we may pick $P_N^* = P_N \setminus \{-\pi\}$ so that P_N^* is a set of test values for P_N.

On the other hand, $g \diamond f$ is the convolution of g and f, which was defined pointwise via Lebesgue integration.

Proof

For a fixed $\theta_0 \in \mathbb{R}$, we may define $\gamma_{\theta_0} : \mathbb{R} \to \mathbb{K}$ as

$$\gamma_{\theta_0}(s) = g(s) f(\theta_0 - s), \quad s \in \mathbb{R}.$$

Since both g and f are continuous and 2π-periodic, γ_{θ_0} is also continuous and 2π-periodic. Thus both sides are bounded and Riemann integrable on $[-\pi, \pi)$. By Theorem 40, we have

$$\begin{aligned} g \diamond f(\theta_0) &= \frac{1}{2\pi} \int_{[-\pi,\pi)} g(s) f(\theta_0 - s) \,\mathrm{dm}(s) \\ &= \frac{1}{2\pi} \int_{[-\pi\pi)} \gamma_{\theta_0}(s) \,\mathrm{dm}(s) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \gamma_{\theta_0}(s) \, ds \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s) f(\theta_0 - s) \, ds. \end{aligned}$$

Since $(\mathbb{C}, |\cdot|)$ is a Banach space, Theorem 3, with the same P_N and P_N^* as defined in our strategy, we have

$$g \diamond f(\theta_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \gamma(s) \, ds = \lim_{N \to \infty} S(\gamma, P_N, P_N^*).$$

Finally,

$$\begin{aligned} &\left\| \Gamma_g^{\circ}(f) - S(\beta, P_N, P_N^*) \right\|_{\sup} \\ &\geq \left| \Gamma_g^{\circ}(f)(\theta_0) - S(\beta, P_N, P_N^*)(\theta_0) \right| \\ &= \left| \Gamma_g^{\circ}(f)(\theta_0) - \sum_{n=1}^{2^N} (\beta(p_n))(\theta_0)(p_n - p_{n-1}) \right| \\ &= \left| \Gamma_g^{\circ}(f)(\theta_0) - \sum_{n=1}^{2^N} (g(p_n) f(\theta_0 - p_n))(p_n - p_{n-1}) \right| \\ &= \left| \Gamma_g^{\circ}(f)(\theta_0) - \sum_{n=1}^{2^N} \gamma_{\theta_0}(s)(p_n - p_{n-1}) \right| \\ &= \left| \Gamma_g^{\circ}(f) - S(\gamma, P_N, P_N^*) \right| \end{aligned}$$

Then, since $\lim_{N\to\infty} \left\| \Gamma_g^\circ(f) - S(\beta, P_N, P_N^*) \right\| = 0$, it follows that

$$\lim_{N\to\infty} \left| \Gamma_g^\circ(f) - S(\gamma, P_N, P_N^*) \right| = 0.$$

Thus

$$\Gamma_g^\circ(f)(\theta_0) = \lim_{N\to\infty} S(\gamma, P_N, P_N^*) = g \diamond f(\theta_0).$$

Since $\theta_0 \in \mathbb{R}$ was arbitrary, we indeed have

$$\Gamma_g^\circ(f) = g \diamond f.$$

□

Theorem 82 (Equivalence of Γ_g and C_g)

Let $g \in \mathcal{C}(\mathbb{T}, \mathbb{C})$ and $[f] \in L_1(\mathbb{T}, \mathbb{C})$. Let Γ_g be as defined before; i.e.

$$\Gamma_g[f] := \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s)\tau_s[f]\, ds,$$

where the integral is a Banach space Riemann integral in $(L_1(\mathbb{T}, \mathbb{C}), \|\cdot\|_1)$. Then

$$\Gamma_g[f] = g * [f] = [g \diamond f] = C_g[f].$$

Proof

By the $\|\cdot\|_1$-density of $[\mathcal{C}(\mathbb{T}, \mathbb{C})]$ in $L_1(\mathbb{T}, \mathbb{C})$, we can make use of Lemma 81. In particular, we can find a sequence $(f_m)_{m=1}^{\infty}$ in $\mathcal{C}(\mathbb{T}, \mathbb{C})$ such that

$$\lim_{n\to\infty} \|[f_m] - [f]\|_1 = 0.$$

Thus, for each $m \geq 1$, we have

$$\Gamma_g[f_m] = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s)\tau_s[f_m]\, ds.$$

Since $f_m \in \mathcal{C}(\mathbb{T}, \mathbb{C})$, for each $m \geq 1$, the map $s \mapsto g(s)\tau_s^\circ(f_m)$ is continuous, thus

$$\Gamma_g^\circ(f_m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s)\tau_s^\circ(f_m)\, ds$$

converges in $(\mathcal{C}(\mathbb{T}, \mathbb{C}), \|\cdot\|_{\sup})$ by the last lemma. In particular, for an appropriate sequence $(P_N)_N$ of partitions of $[-\pi, \pi]$, we have that

$$\Gamma_g^\circ(f_m) = \lim_{N\to\infty} S(\beta_m, P_N, P_N^*),$$

where $\beta_m = g(s)\tau_s^\circ(f_m)$, for $s \in [-\pi, \pi)$. But given any Riemann sum $S(\beta_m, Q, Q^*)$ of the form

$$\sum_{k=1}^{M} \beta_m(q_k^*)(q_k - q_{k-1}) = \sum_{k=1}^{M} g(q_k^*)\tau_{q_k^*}^\circ(f_m)(q_k - q_{k-1})$$

in $\mathcal{C}(\mathbb{T}, \mathbb{C})$, its image in $[\mathcal{C}(\mathbb{T}, \mathbb{C})]$ is

$$[S(\beta_m, Q, Q^*)] = \sum_{k=1}^{M} g(q_k^*)\tau_{q_k^*}[f_m](q_k - q_{k-1}).$$

Since $h \mapsto [h]$ from $(\mathcal{C}(\mathbb{T}, \mathbb{C}), \|\cdot\|_{\sup})$ to $([\mathcal{C}(\mathbb{T}, \mathbb{C})], \|\cdot\|_\infty)$ is a bijective linear isometry, the image of $\Gamma_g^\circ(f_m)$ under this map is

$$[\Gamma_g^\circ(f_m)] = \lim_{N\to\infty} [S(\beta_m, P_N, P_N^*)],$$

and this convergence is wrt the $\|\cdot\|_\infty$-norm.

On the other hand, by the definition of each $[S(\beta_m, P_N, P_N^*)]$, we have, precisely,

$$\lim_{N\to\infty} [S(\beta_m, P_N, P_N^*)] = \Gamma_g([f_m]) \in ([\mathcal{C}(\mathbb{T}, \mathbb{C})], \|\cdot\|_\infty),$$

thus

$$[\Gamma_g^\circ(f_m)] = \Gamma_g[f_m], \quad m \geq 1.$$

Now

$$[S(\beta_m, P_N, P_N^*)] \in [\mathcal{C}(\mathbb{T}, \mathbb{C})] \subseteq L_1(\mathbb{T}, \mathbb{C})$$

and

$$\Gamma_g[f_m] \in [\mathcal{C}(\mathbb{T}, \mathbb{C})] \subseteq L_1(\mathbb{T}, \mathbb{C}).$$

Since $[\mathcal{C}(\mathbb{T}, \mathbb{C})]$ is a homogeneous Banach space, $\|[h]\|_1 \leq \|[h]\|_\infty$ for all $[h] \in [\mathcal{C}(\mathbb{T}, \mathbb{C})]$. Thus

$$\begin{aligned} 0 &\leq \lim_{N\to\infty} \left\| [\Gamma_g^\circ(f_m)] - [S(\beta_m, P_N, P_N^*)] \right\|_1 \\ &\leq \lim_{N\to\infty} \left\| [\Gamma_g^\circ(f_m)] - [S(\beta_m, P_N, P_N^*)] \right\|_\infty = 0, \end{aligned}$$

and so

$$\Gamma_g[f_m] = [\Gamma_g^\circ(f_m)] = \lim_{N\to\infty} [S(\beta_m, P_N, P_N^*)],$$

where the convergence happens in $(L_1(\mathbb{T}, \mathbb{C}), \|\cdot\|_1)$.

Step 1 WTS $\forall m \geq 1, \Gamma_g[f_m] = g * [f_m]$. By Lemma 81, $\forall m \geq 1$, we have $\Gamma_g^\circ(f_m) = g \diamond f_m$. Thus

$$\Gamma_g[f_m] = [\Gamma_g^\circ(f_m)] = [g \diamond f_m] = g * [f_m]$$

for $m \geq 1$.

Step 2 WTS $g * [f] = \lim_{m\to\infty} g * [f_m]$ in $(L_1(\mathbb{T}, \mathbb{C}), \|\cdot\|_1)$. One way we can show this is by realizing that we want

$$0 = \lim_{m\to\infty} (g * [f_m] - g * [f]) = \lim_{m\to\infty} [g \diamond f_m - g \diamond f],$$

and for $\theta \in \mathbb{R}$,

$$\begin{aligned}(g \diamond f_m - g \diamond f)(\theta) &= \frac{1}{2\pi} \int_{[-\pi,\pi)} g(s)(f_m(\theta - s) - f(\theta - s)) \, \mathrm{d}m(s) \\ &= g \diamond (f_m - f)(\theta).\end{aligned}$$

As noted after Definition 50, we have

$$|g \diamond (f_m - f)(\theta)| \leq \|g\|_{\sup} \|[f_m] - [f]\|_1 .$$

Thus for $m \geq 1$,

$$\begin{aligned}\|g * [f_m] - g * [f]\|_1 &= \|g * [f_m - f]\|_1 \\ &= \frac{1}{2\pi} \int_{[-\pi,\pi)} |g \diamond (f_m - f)(\theta)| \, \mathrm{d}m(\theta) \\ &\leq \frac{1}{2\pi} \int_{[-\pi,\pi)} \|g\|_{\sup} \|[f_m] - [f]\|_1 \, \mathrm{d}m(\theta) \\ &= \|g\|_{\sup} \|[f_m] - [f]\|_1 .\end{aligned}$$

Since

$$\lim_{m\to\infty} \|[f_m] - [f]\|_1 = 0,$$

[1] it follows that

$$g * [f] = \lim_{m\to\infty} g * [f_m]$$

in $(L_1(\mathbb{T}, \mathbb{C}), \|\cdot\|_1)$.

[1] I bet you forgot this! :P

Step 3 WTS $\Gamma_g[f] = \lim_{m\to\infty} \Gamma_g[f_m]$ in $(L_1(\mathbb{T}, \mathbb{C}), \|\cdot\|_1)$. We see that by the properties of a homogeneous Banach space, we have

$$\begin{aligned} \left\|\Gamma_g[f] - \Gamma_g[f_m]\right\|_1 &= \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s) \tau_s[f - f_m] \, ds \right\|_1 \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s) \left\|\tau_s[f - f_m]\right\|_1 ds \\ &\leq \|g\|_{\sup} \left\|\tau_s[f - f_m]\right\|_1 \\ &= \|g\|_{\sup} \left\|[f] - [f_m]\right\|_1 . \end{aligned}$$

As before, it follows that

$$\Gamma_g[f] = \lim_{m\to\infty} \Gamma_g[f_m] \quad \in (L_1(\mathbb{T}, \mathbb{C}), \|\cdot\|_1).$$

Step 4 Finally, we see that

$$\Gamma_g[f] = \lim_{m\to\infty} \Gamma_g[f_m] = \lim_{m\to\infty} g * [f_m] = g * [f]. \qquad \square$$

Viewing Γ_g as a map from $L_1(\mathbb{T}, \mathbb{C})$ onto itself, we finally conclude our gruesome path into showing that $\Gamma_g = C_g$. Thus, our 2 "notions" of "convolutions" agree. In fact, when $g \in \mathcal{C}(\mathbb{T}, \mathbb{C})$, we may define

$$\begin{aligned} \Gamma_g^{\mathfrak{B}} : \mathfrak{B} &\to \mathfrak{B} \\ [f] &\mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s) \tau_s[f] \, ds \end{aligned}$$

as a map on any homogeneous Banach space $\mathfrak{B}$ over $\mathbb{T}$. Furthermore, $C_g[f] = [g \diamond f] \in \mathfrak{B}$.

Let us show that the above function always agree with convolution.

Theorem 83 (Riemannian Version of Convolution on Homogeneous Banach Spaces)

Let $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$ *be a homogeneous Banach space over* $\mathbb{T}$, $[f] \in \mathfrak{B}$, *and* $g \in \mathcal{C}(\mathbb{T}, \mathbb{C})$. *Then*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} g(s) \tau_s[f] \, ds$$

converges in $\mathfrak{B}$. *Furthermore,*

1. $$g * [f] = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s)\tau_s[f] \, ds,$$

   where $g * [f] = [g \diamond f]$, and where for all $\theta \in \mathbb{R}$,

   $$g \diamond f(\theta) = \frac{1}{2\pi} \int_{[-\pi,\pi)} g(s) f(\theta - s) \, \mathrm{dm}(s).$$

   That is, $\Gamma_g^{\mathfrak{B}}[f] = g * [f]$.

2. *We also have*

   $$\| g * [f] \|_{\mathfrak{B}} \leq \nu_1(g) \, \| [f] \|_{\mathfrak{B}} \, .$$

Proof

1. Since $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$ is a Banach space, and for $[f] \in \mathfrak{B}$, the function $\beta : \mathbb{R} \to \mathfrak{B}$ such that

   $$\beta(s) := g(s)\tau_s[f]$$

   is continuous, by Theorem 3,

   $$\Gamma_g^{\mathfrak{B}}[f] := \frac{1}{2\pi} \int_{-\pi}^{\pi} \beta(s) \, ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s)\tau_s[f] \, ds$$

   exists in $\mathfrak{B}$.

   As before, wlog wma $P_N \in \mathcal{P}([-\pi, \pi))$ is a regular partition into 2^N subintervals of equal length, and if we set $P_N^* = P_N \setminus \{\pi\}$ as the corresponding set of test values of P_N, then

   $$\lim_{N \to \infty} \left\| \Gamma_g^{\mathfrak{B}}[f] - S(\beta, P_N, P_N^*) \right\|_{\mathfrak{B}} = 0.$$

   Since $\|[h]\|_1 \leq \|[h]\|_{\mathfrak{B}}$ for all $[h] \in \mathfrak{B}$, since $\mathfrak{B}$ is a homogeneous Banach space. Thus

   $$\lim_{N \to \infty} \left\| \Gamma_g^{\mathfrak{B}}[f] - S(\beta, P_N, P_N^*) \right\|_1 = 0.$$

   Hence

   $$\Gamma_g^{\mathfrak{B}}[f] = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s)\tau_s[f] \, ds$$

   in $(L_1(\mathbb{T}, \mathbb{C}), \|\cdot\|_1)$. Phrased differently, we have $\Gamma_g^{\mathfrak{B}}[f] = \Gamma_g^{L_1(\mathbb{T},\mathbb{C})}[f]$.

It follows by Theorem 82 that

$$\Gamma_f^{\mathfrak{B}}[f] = \Gamma_g^{L_1(\mathbb{T},\mathbb{C})}[f] = g * [f].$$

2. Recall that given any homogeneous Banach space over $\mathbb{T}$, we defined the continuous map $\Psi_{[f]} : \mathbb{R} - > \mathfrak{B}$ such that $\Psi_{[f]}(s) = \tau_s[f]$. Notice that since $\mathfrak{B}$ is a homogeneous Banach space,

$$\left\| \Psi_{[f]}(s) \right\|_{\mathfrak{B}} = \| \tau_s[f] \|_{\mathfrak{B}} = \| [f] \|_{\mathfrak{B}}.$$

Thus, observe that

$$\begin{aligned} \| g * [f] \|_{\mathfrak{B}} &= \frac{1}{2\pi} \left\| \int_{-\pi}^{\pi} g(s) \tau_s[f] \, ds \right\|_{\mathfrak{B}} \\ &= \frac{1}{2\pi} \left\| \int_{-\pi}^{\pi} g(s) \Psi_{[f]}(s) \, ds \right\|_{\mathfrak{B}} \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{;p} |g(s)| \left\| \Psi_{[f]}(s) \right\|_{\mathfrak{B}} ds \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(s)| \, \| [f] \|_{\mathfrak{B}} \, ds \\ &= \| [f] \|_{\mathfrak{B}} \, \nu_1(g). \end{aligned}$$

□

Remark 19.1.1

The first result in Theorem 83 is stronger than it seems. In Proposition 80, we showed that if $g \in \mathcal{C}(\mathbb{T}, \mathbb{C})$ and $f \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$, then $g \diamond f \in \mathcal{C}(\mathbb{T}, \mathbb{C})$, and so $g * [f] := [g \diamond f] \in [\mathcal{C}(\mathbb{T}, \mathbb{C})]$. Then why is $g * [f] \in \mathfrak{B}$? There is no reason why $\mathfrak{B}$ should contain all continuous functions, although it does contain all trigonometric functions. What we have shown is that $g * [f] \in \mathfrak{B}$ even if $\mathfrak{B}$ does not contain $[\mathcal{C}(\mathbb{T}, \mathbb{C})]$. In other words, convolutions (at least by a continuous function) keeps us in this smaller space $\mathfrak{B}$.

Unfortunately, I am running out of time for studying everything carefully. I shall be skipping proofs while covering for everything else. I will come back to adding proofs when time permits.

Theorem 84 (Convolution as a Normalizer)

Let $g \in \mathcal{C}(\mathbb{T}, \mathbb{C})$, and let

$$C_g : ([\mathcal{C}(\mathbb{T}, \mathbb{C})], \|\cdot\|_\infty) \to ([\mathcal{C}(\mathbb{T}, \mathbb{C})], \|\cdot\|_\infty),$$

as defined in Definition 52, corresponding to g, so that $C_g[h] = g * [h]$ by Theorem 82. Then $\|C_g\| = \nu_1(g) = \|[g]\|_1$.

Proof

To be added □

20 Lecture 20 Jul 18th 2019

20.1 Convolution (Continued 3)

Let us establish a similar result for convolution by a continuous function g acting on $L_1(\mathbb{T}, \mathbb{C})$.

Theorem 85 (Convolution Operator for g on $L_1(\mathbb{T}, \mathbb{C})$)

Let $g \in \mathcal{C}(\mathbb{T}, \mathbb{C})$, and

$$C_g : L_1(\mathbb{T}, \mathbb{C}) \to L_1(\mathbb{T}, \mathbb{C})$$

be the convolution operator corresponding to g, so that $C_g[f] = g * [f]$. Then $\|C_g\| = \nu_1(g) = \|[g]\|_1$.

Proof

To be added □

We shall next explore the connection between convolution operators and convergence of Fourier series.

20.2 The Dirichlet Kernel

Recall that given $[f] \in L_2(\mathbb{T}, \mathbb{C})$, the sequence $(\Delta_N([f]))_{N=1}^{\infty}$ of partial sums of the Fourier series of $[f]$ converges in the $\|\cdot\|_2$-norm to $[f]$.

We want to see how far we can extend the same result for $[f] \in$

$L_1(\mathbb{T}, \mathbb{C})$. Our little (not so little) impasse into convolution is actually somewhat important to the so-called Dirichlet kernel, of which we shall see below, and why that's important is also what we shall quite immediately see.

Definition 54 (Dirichlet Kernel of Order N)

For each $n \in \mathbb{Z}$, recall that $\xi_n \in \mathcal{C}(\mathbb{T}, \mathbb{C})$ is the function $\xi_n(\theta) = e^{in\theta}$. For $N \geq 1$, we define the ***Dirichlet kernel of order*** N *as*

$$D_N = \sum_{n=-N}^{N} \xi_n.$$

Note 20.2.1

Again, the word 'kernel' has nothing to do with the null space of any linear map.

Let $f \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$. For each $N \geq 1$, define

$$\Delta_N^\circ(f) := \sum_{n=-N}^{N} \alpha_n^{[f]} \xi_n = \sum_{n=-N}^{N} \hat{f}(n) \xi_n.$$

It is clear that $\Delta_N^\circ(f) \in \mathcal{C}(\mathbb{T}, \mathbb{C})$, since it is a finite linear combination of $\{\xi_n\}_{n=-N}^{N} \subseteq \mathcal{C}(\mathbb{T}, \mathbb{C})$.

If $f = g$ a.e. on $\mathbb{R}$, we saw that $\alpha_n^{[f]} = \alpha_n^{[g]}$ for all $n \in \mathbb{Z}$, which then $\Delta_N^\circ(f) = \Delta_N^\circ(g)$ for $N \geq 1$. Thus, we may define

$$\Delta_N([f]) = [\Delta_N^\circ(f)], \quad N \geq 1.$$

Hence, $\Delta_N([f])$ is the N^{th} partial sum of the Fourier series of $[f]$. In the case of $[f] \in L_2(\mathbb{T}, \mathbb{C})$, this definition coincides with our previous definition.

For $N \geq 1$, $f \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$ and $\theta \in \mathbb{R}$, we have

$$\Delta_N^\circ(f)(\theta) = \sum_{n=-N}^{N} \alpha_n^{[f]} \xi_n$$

$$
\begin{aligned}
&= \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{[-\pi,\pi)} f(s)\overline{\xi_n}(s) \, \mathrm{dm}(s) \right) \xi_n(\theta) \\
&= \sum_{n=-N}^{N} \frac{1}{2\pi} \int_{[-\pi,\pi)} f(s) e^{in(\theta - s)} \, \mathrm{dm}(s) \\
&= \sum_{n=-N}^{N} \frac{1}{2\pi} \int_{[-\pi,\pi)} f(\theta - s) e^{ins} \, \mathrm{dm}(s) \quad \because \text{Lemma 79} \\
&= \frac{1}{2\pi} \int_{[-\pi,\pi)} \sum_{n=-N}^{N} f(\theta - s) e^{ins} \, \mathrm{dm}(s) \\
&= \frac{1}{2\pi} \int_{[-\pi,\pi)} D_N(s) f(\theta - s) \, \mathrm{dm}(s) \\
&= (D_N \diamond f)(\theta).
\end{aligned}
$$

Thus $\Delta_N^\circ(f) = D_N \diamond f$, or

$$\Delta_N([f]) = D_N * [f] = C_{D_N}([f]), \quad N \geq 1.$$

We expressed the N^{th} partial sum of the Fourier series of $[f] \in L_1(\mathbb{T}, \mathbb{C})$ as the convolution of the Dirichlet kernel D_N of order N with $[f]$. [1]

The question of whether or not these partial sums converge to $[f]$ in $L_1(\mathbb{T}, \mathbb{C})$ is now a question of whether or not $\lim_{N\to\infty} C_{D_N}([f]) = [f]$ in $L_1(\mathbb{T}, \mathbb{C})$.

[1] Our sweats and tears ploughing through the convoluted lands of convolutions is not confounded!

Theorem 86 (Properties of the Dirichlet Kernel)

Let $N \geq 1$ be an integer and D_N be the Dirichlet kernel of order N. Then

1. $D_N(-\theta) = D_N(\theta) \in \mathbb{R}$ *for all* $\theta \in \mathbb{R}$*;*
2. $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(\theta) \, d\theta = 1$*;*
3. *For* $0 \neq \theta \in [-\pi, \pi)$*,*

$$D_N(\theta) = \frac{\sin(\left(N + \frac{1}{2}\right) \theta)}{\sin\left(\frac{1}{2}\theta\right)}.$$

   Also, $D_N(0) = 2N + 1$.

4. $\|[D_N]\|_1 = \nu_1(D_N) \geq \frac{4}{\pi^2} \sum_{n=1}^{N} \frac{1}{n}$.

Proof

To be added □

On the right are some graphs of D_N, particularly for D_2, D_5 and D_{10}.

Two things worth noticing here are that

- the amplitude of the function is increasing near 0; this is clear from D_n being continuous and $D_N(0) = 2N + 1$ for $N \geq 1$; and
- each D_N has lots of fluctuations between positive and negative values, which accounts for the fact that the integrals of D_N are bounded, while the integrals of $|D_N|$ are not.

The next result follows from Theorem 83 and Theorem 84, and along with the divergence of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ as we let $N \to \infty$ for the 4th result in Theorem 86.

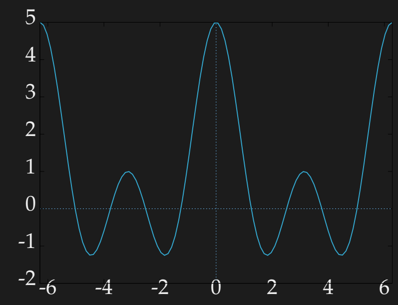

Figure 20.1: Graph of D_2

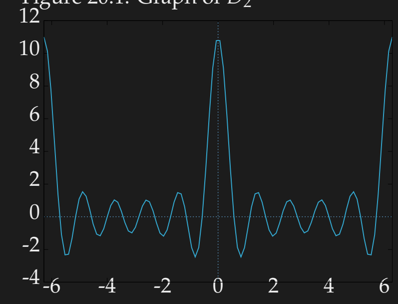

Figure 20.2: Graph of D_5

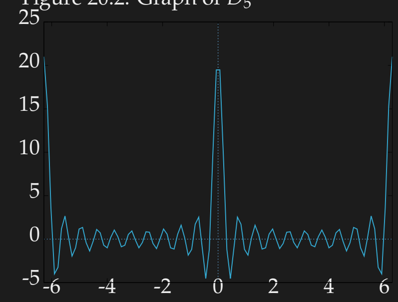

Figure 20.3: Graph of D_{10}

Corollary 87 (Unboundedness of Convolution Operators for the Dirichlet Kernel)

For each $N \geq 1$, let D_N denote the Dirichlet kernel of order N.

1. *If $C_{D_N} \in \mathcal{B}([\mathcal{C}(\mathbb{T}, \mathbb{C})], \|\cdot\|_\infty)$ is the convolution operator corresponding to D_N, for $N \geq 1$, then*
$$\lim_{N\to\infty} \|C_{D_N}\| = \infty.$$
2. *If $C_{D_N} \in \mathcal{B}(L_1(\mathbb{T}, \mathbb{C}), \|\cdot\|_1)$ is the convolution operator corresponding to D_N, for $N \geq 1$, then*
$$\lim_{N\to\infty} \|C_{D_N}\| = \infty.$$

We are already seeing some bad signs of things not working out nicely. Let's push a little bit further. To exploit the connection between the Dirichlet kernel and convolution, we require a few results from real analysis.

Definition 55 (Nowhere Dense)

*Let (X, d) be a metric space and $H \subseteq X$. We say that H is **nowhere dense** (or **meager**, or **thin**) if $G := X \setminus \overline{H}$ is dense in X. In other words, the interior of $\overline{H}$ is empty.*

Example 20.2.1

We usually think of nowhere dense subsets of metric spaces as being "small", as the alternate terminologies "meager" and "thin" suggest.

1. The set $H = \mathbb{Z}$ is nowhere dense in $\mathbb{R}$; which is easily verifiable.
2. The Cantor set C is nowhere dense in $X = [0, 1]$, equipped with the standard metric inherited from $\mathbb{R}$.
3. The set $H = \mathbb{Q}$ of rational numbers is **not** nowhere dense in $\mathbb{R}$, since $X \setminus \overline{H} = \mathbb{R} \setminus \mathbb{R} = \emptyset$.

Definition 56 (First and Second Category)

*We say that a subset H of a metric space (X, d) is of the **first category** in (X, d) if there exists a sequence $(F_n)_n$ of closed, nowhere dense sets in X such that*

$$H \subseteq \bigcup_{n=1}^{\infty} F_n.$$

*Otherwise, we say that H is of **second category**.*

The reader should be familiar with the following result.

Definition 57 (Baire Category Theorem)

A complete metric space (X, d) is of the second category. That is, X is not a countable union of closed, nowhere dense sets in X.

21 Lecture 21 Jul 23rd 2019

21.1 The Dirichlet Kernel (Continued)

The second result which we shall require is the following.

Theorem 88 (Banach-Steinhaus Theorem (aka The Uniform Boundedness Principle))

Let (X, d) *be a complete metric space and* $\emptyset \neq \mathcal{F} \subseteq \mathcal{C}(X, \mathbb{R})$*. Suppose that* $\forall x \in X$, $\exists \kappa_x > 0$ *a constant such that*

$$|f(x)| \leq \kappa_x, \quad f \in \mathcal{F}.$$

Then there exists an open set $G \subseteq X$ *and* $\kappa > 0$ *such that*

$$|f(x)| \leq \kappa, \quad x \in G, \;\; f \in \mathcal{F}.$$

Proof

To be added □

There is a stronger version of the Banach-Steinhaus Theorem that applies to linear operators in Banach spaces.

Theorem 89 (Banach-Steinhaus Theorem for Operators (aka The Uniform Boundedness Principle for Operators))

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ and $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ be Banach spaces, and suppose that $\emptyset \neq \mathcal{F} \subseteq \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. Let $H \subseteq \mathfrak{X}$ be a subset of the second category in $\mathfrak{X}$, and suppose that for each $x \in \mathcal{H}$, there exists a constant $\kappa_x > 0$ such that

$$\|Tx\|_{\mathfrak{Y}} \leq \kappa_x, \quad T \in \mathcal{F}.$$

Then $\mathcal{F}$ is bounded; that is

$$\sup_{T \in \mathcal{F}} \|T\| < \infty.$$

Proof

To be added □

Corollary 90 (Sparcity of Boundedness of Unbounded Sequences of Bounded Functions between Banach Spaces)

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ and $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ be Banach spaces, and let $(T_n)_{n=1}^{\infty}$ be an unbounded sequence in $\mathcal{C}(\mathfrak{X}, \mathfrak{Y})$, i.e. $\sup_{n \geq 1} \|T_n\| = \infty$.

Let $H = \{x \in \mathfrak{X} : \sup_{n \geq 1} \|T_n x\| < \infty\}$. Then H is of the first category in $\mathfrak{X}$, and $J := \mathfrak{X} \setminus H$ is of the second category.

Proof

To be added □

Remark 21.1.1 (★ Further implications of Corollary 90)

The statement that $\sup_{n \geq 1} \|T_n\| = \infty$ is the statement that for each $n \geq 1$, there exists $x_n \in \mathfrak{X}$ with $\|x_n\|_{\mathfrak{X}} = 1$ such that

$$\lim_{n \to \infty} \|T_n x_n\|_{\mathfrak{Y}} = \infty.$$

In the first place, it is not clear that there should exist any $x \in \mathfrak{X}$ such that

$$\lim_{n \to \infty} \|T_n x\|_{\mathfrak{Y}} = \infty.$$

The above corollary not only says that such a vector $x \in \mathfrak{X}$ exists; it asserts that this is true for a 'very large' set of x's, in the sense that the set H of x's for which it fails is a set of the first category in $\mathfrak{X}$.

We are now ready to answer the question of whether or not the partial sums of the Fourier series of an element $[f] \in L_1(\mathbb{T}, \mathbb{C})$ necessarily converge to $[f]$ in the $\|\cdot\|_1$-norm. We shall see that by Corollary 90, this convergence **almost never happens**. [1] Furthermore, the same argument shows that this is also the case for $[f] \in [\mathcal{C}(\mathbb{T}, \mathbb{C})]$ in the $\|\cdot\|_\infty$.

[1] We use the phrase 'almost never' to mean the notion of first category, not measure zero.

Theorem 91 (The *unbearable lousiness* of being a Dirichlet Kernel)

1. *Let*

$$\mathfrak{K}_\infty := \{[f] \in [\mathcal{C}(\mathbb{T}, \mathbb{C})] : [f] = \lim_{N \to \infty} \Delta_N [f] \in ([\mathcal{C}(\mathbb{T}, \mathbb{C})], \|\cdot\|_\infty)\}.$$

Then $\mathfrak{K}_\infty$ is a set of the first category in $([\mathcal{C}(\mathbb{T}, \mathbb{C})], \|\cdot\|_\infty)$, whose complement $[\mathcal{C}(\mathbb{T}, \mathbb{C})] \setminus \mathfrak{K}_\infty$ is a set of the second category.

2. *Let*

$$\mathfrak{K}_1 := \{[f] \in L_1(\mathbb{T}, \mathbb{C}) : [f] = \lim_{N \to \infty} \Delta_N [f] \in (L_1(\mathbb{T}, \mathbb{C}), \|\cdot\|_1)\}.$$

Then $\mathfrak{K}_1$ is a set of the first category in $(L_1(\mathbb{T}, \mathbb{C}), \|\cdot\|_1)$, whose complement $L_1(\mathbb{T}, \mathbb{C}) \setminus \mathfrak{K}_1$ is a set of the second category.

Proof

To be added □

We have entered the darkest days of our course, and almost at the very end of our journey. If this were a novel, I bet readers would be crying over how their favorite hero has fallen, and the fragility of our other heroes. But there is some hope in the face of this despair.

21.2 The Féjer Kernel

Definition 58 (N^{th}-Cesàro mean)

Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ be a Banach space, and $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathfrak{X}$. The ***N^{th}-Cesàro mean*** *of the sequence is defined as*

$$\sigma_N := \frac{1}{N} (x_0 + x_1 + \ldots + x_{N-1}),$$

for $N \geq 1$.

Proposition 92 (Convergent Sequences have Convergent Cesàro Means)

Suppose that $\mathfrak{X}$ is a Banach space and $(x_n)_{n=0}^{\infty}$ is a sequence in $\mathfrak{X}$. Let $(\sigma_N)_{N=1}^{\infty}$ denote the sequence of Cesàro means of $(x_n)_{n=1}^{\infty}$. If $x = \lim_{n\to\infty} x_n$ exists, then

$$x = \lim_{N\to\infty} \sigma_N.$$

Exercise 21.2.1

Prove Proposition 92.

Remark 21.2.1

The converse of Proposition 92 is false. Let $(x_n)_{n=1}^{\infty} = ((-1)^n)_{n=1}^{\infty}$. Then

$$(x_0, x_1, x_2, \ldots) = (1, -1, 1, \ldots).$$

We see that $|\sigma_N| \leq \frac{1}{N}$, which then $\lim_{N\to\infty} \sigma_N = 0$, but $\lim_{n\to\infty} x_n$ does not exist.

Definition 59 (N^{th}-Cesàro sum and the Féjer kernel of order N)

*Let $f \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$. The **$N^{th}$-Cesàro sum** of the Fourier series of f is the N^{th}-Cesàro mean of the sequence $(\Delta_n^\circ(f))_{n=0}^\infty$. Thus*

$$\sigma_N^\circ(f) = \frac{1}{N}(D_0 \diamond f + D_1 \diamond f + \ldots D_{N-1} \diamond f) = F_N \diamond f,$$

*where $F_N := \frac{1}{N}(D_0 + D_1 + \ldots + D_{N-1})$ is called the **Féjer kernel of order N**.*

*We also define the **N^{th}-Cesàro sum** of the Fourier series of $[f] \in L_1(\mathbb{T}, \mathbb{C})$ as the N^{th}-Cesàro mean of the sequence $(\Delta_n[f])_{n=0}^\infty$, namely*

$$\begin{aligned}\sigma_N[f] &:= \frac{1}{N}(D_0 * [f] + D_1 * [f] + \ldots + D_{N-1} * [f]) \\ &= F_N * [f] = [F_N \diamond f] = [\sigma_N^\circ(f)].\end{aligned}$$

Remark 21.2.2

$D_n \in \mathcal{C}(\mathbb{T}, \mathbb{C})$ for all $n \geq 0$ implies that $F_N \in \mathcal{C}(\mathbb{T}, \mathbb{C})$ for all $N \geq 1$. By Proposition 80, it follows that $\sigma_N^\circ(f) \in \mathcal{C}(\mathbb{T}, \mathbb{C}) \subseteq \mathcal{L}_1(\mathbb{T}, \mathbb{C})$ for every $f \in \mathcal{L}_1(\mathbb{T}, \mathbb{C})$.

Furthermore, $\forall \theta \in \mathbb{R}$,

$$\begin{aligned}\sigma_N^\circ(f)(\theta) &= \frac{1}{2\pi}\int_{[-\pi,\pi)} F_N(s) f(\theta - s)\,\mathrm{dm}(s) \\ &= \frac{1}{2\pi}\int_{[-\pi,\pi)} F_N(\theta - s) f(s)\,\mathrm{dm}(s).\end{aligned}$$

By Theorem 83, $F_N \in \mathcal{C}(\mathbb{T}, \mathbb{C})$ implies that for every homogeneous Banach algebra $\mathfrak{B}$ and $[f] \in \mathfrak{B}$, we have

$$\sigma_N[f] = F_N * [f] \in \mathfrak{B}.$$

In particular, $\forall [f] \in L_p(\mathbb{T}, \mathbb{C})$,

$$\sigma_N[f] = F_N * [f] \in L_p(\mathbb{T}, \mathbb{C}).$$

Theorem 93 (Properties of the Féjer Kernel)

For each $N \in \mathbb{N} \setminus \{0\}$,

1. *F_N is a 2π-periodic, even, continuous function;*

2. *If $0 \neq \theta \in [-\pi, \pi)$, then*

$$F_N(\theta) = \frac{1}{N}\left(\frac{1-\cos(N\theta)}{1-\cos(\theta)}\right) = \frac{1}{N}\left(\frac{\sin\left(\frac{N}{2}\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}\right)^2,$$

while $F_N(0) = N$. In particular, $F_N(\theta) \geq 0$ for all $\theta \in \mathbb{R}$;

3. $$\nu_1(F_N) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |F_N(\theta)|\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(\theta)\, d\theta = 1.$$

4. *For all $0 < \delta \leq \pi$,*

$$\lim_{N\to\infty}\left(\int_{-\pi}^{-\delta} |F_N(\theta)|\, d\theta + \int_{\delta}^{\pi} |F_N(\theta)|\, d\theta\right) = 0; \text{ and}$$

5. *For $0 < |\theta| < \pi$,*

$$0 \leq F_N(\theta) \leq \frac{\pi^2}{N\theta^2}.$$

Proof

To be added □

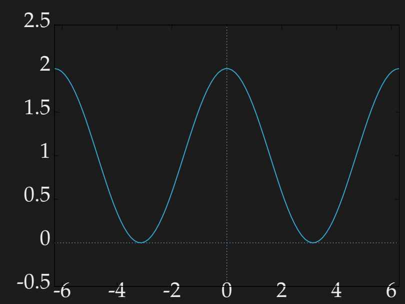

Figure 21.1: Graph of F_2

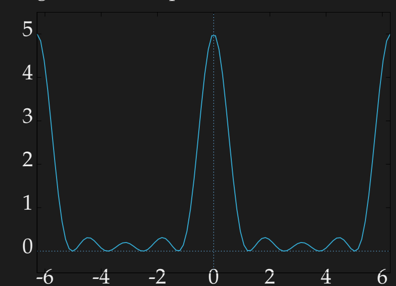

Figure 21.2: Graph of F_5

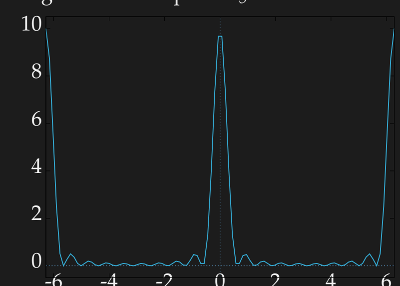

Figure 21.3: Graph of F_{10}

To the right are some graphs of F_N, particularly for F_2, F_5 and F_{10}.

Two things worth noticing about them are:

- the amplitude of the function is increasing near 0; this is as we've seen for the Dirichlet kernel, where F_N is continuous and $F_N(0) = N$ for $N \geq 1$; and
- for each $\delta > 0$, the functions become uniformly close to 0 when $\delta < |\theta| < \pi$.

Let us pull out D_5 and F_5 for comparison.

For both D_5 and F_5, their respective 1-norms from $-\pi$ to π sums to 1. For D_5, we see that there are many regions where the function

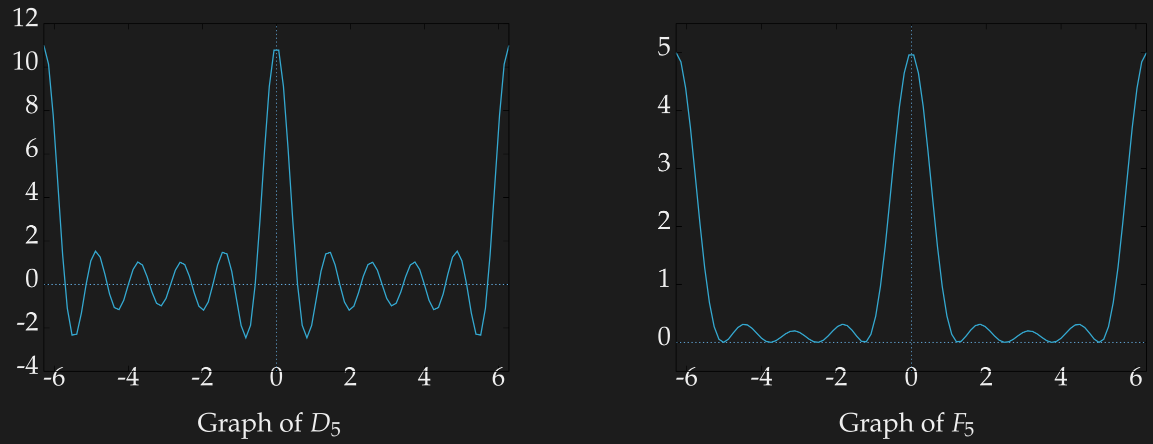

Figure 21.4: Comparing D_5 and F_5

is negative, whereas F_5 is always positive. Furthermore, F_N has the property that

$$\lim_{N\to\infty}\left(\int_{-\pi}^{-\delta}|F_N(\theta)|\,d\theta+\int_{\delta}^{\pi}|F_N(\theta)|\,d\theta\right)=0,$$

of which D_N does not.

22 Lecture 22 Jul 25th 2019

22.1 The Féjer Kernel (Continued)

Definition 60 (Summability Kernel)

*A **summability kernel** is a sequence $(k_n)_{n=1}^{\infty}$ of 2π-periodic, continuous,* [1] *complex-valued functions $\mathbb{R}$ satisfying:*

1. $\frac{1}{2\pi}\int_{-\pi}^{\pi} k_n = 1$ *for all* $n \geq 1$;
2. $\sup_{n \geq 1} \nu_1(k_n) = \sup_{n \geq 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} |k_n| < \infty$; *and*
3. *for all* $0 < \delta \leq \pi$,

$$\lim_{n \to \infty} \left(\int_{-\pi}^{-\delta} |k_n| + \int_{\delta}^{\pi} |k_n| \right) = 0.$$

*If we further have $k_n \geq 0$ for all $n \geq 1$, we say that $(k_n)_{n=1}^{\infty}$ is a **positive summability kernel**.*

[1] A summability kernel can be more general than a continuous sequence, but for our purposes, this is sufficient.

Theorem 94 (Féjer kernel as a Positive Summability Kernel)

The Féjer kernel $(F_N)_{N=1}^{\infty}$ is a positive summability kernel.

Proof

Theorem 93 proves exactly this. □

Example 22.1.1 (Other examples of positive summability kernels)

1. For each $n \in \mathbb{N} \setminus \{0\}$, consider the piecewise linear function

$$k_n^\bullet : [-\pi, \pi) \to \mathbb{R}$$
$$\theta \mapsto \begin{cases} 0 & \theta \in \left[-\pi, -\frac{1}{n}\right] \cup \left[\frac{1}{n}, \pi\right) \\ n + n^2\theta & \theta \in \left(-\frac{1}{n}, 0\right] \\ n - n^2\theta & \theta \in \left(0, \frac{1}{n}\right) \end{cases}.$$

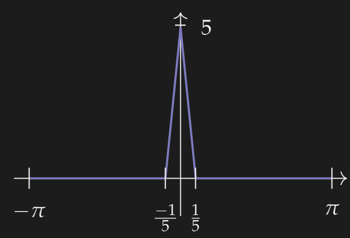

Figure 22.1: Graph of $k_5^\bullet$

For $n \in \mathbb{N} \setminus \{0\}$, let k_n be the 2π-periodic function on $\mathbb{R}$ whose restriction to the interval $[-\pi, \pi)$ coincides with $k_n^\bullet$. Then $(k_n)_{n=1}^\infty$ is a positive summability kernel.

2. For each $n \in \mathbb{N} \setminus \{0\}$, consider the piecewise linear function

$$r_n^\bullet : [-\pi, \pi) \to \mathbb{R}$$
$$\theta \mapsto \begin{cases} 0 & \theta \in [-\pi, 0] \cup \left[\frac{2}{n}, \pi\right) \\ n^2\theta & \theta \in \left(0, \frac{1}{n}\right] \\ n - n^2\left(\theta - \frac{1}{n}\right) & \theta \in \left(\frac{1}{n}, \frac{2}{n}\right) \end{cases}.$$

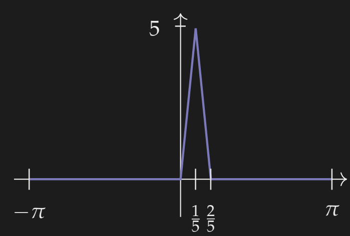

Figure 22.2: Graph of $r_5^\bullet$

For $n \in \mathbb{N} \setminus \{0\}$, let r_n be the 2π-periodic function on $\mathbb{R}$ whose restriction to the interval $[-\pi, \pi)$ coincides with $k_n^\bullet$. Then $(r_n)_{n=1}^\infty$ is a positive summability kernel.

Theorem 95 (Summability kernels convolved with functions in Homogeneous Banach Spaces)

Let $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$ be a homogeneous Banach space over $\mathbb{T}$ and $(k_n)_{n=1}^\infty$ be a summability kernel. If $[f] \in \mathfrak{B}$, then

$$\lim_{n \to \infty} \|k_n * [f] - [f]\|_{\mathfrak{B}} = 0,$$

and so $[f] = \lim_{n\to\infty} k_n * [f]$ in $\mathfrak{B}$.

Proof

To be added □

Corollary 96 (Reconcilation of the Cesàro sums to the Original Function)

1. *For each* $f \in (\mathcal{C}(\mathbb{T}, \mathbb{C}), \|\cdot\|_{\sup})$,

$$\lim_{N \to \infty} \sigma_N^{\circ}(f) = f.$$

2. *Let* $1 \leq p < \infty$. *For each* $[g] \in (L_p(\mathbb{T}, \mathbb{C}), \|\cdot\|_p)$,

$$\lim_{N \to \infty} \sigma_N[g] = [g].$$

Proof

To be added □

We can now show that the Fourier coefficients of functions of $\mathcal{L}_p(\mathbb{T}, \mathbb{C})$ completely determine themselves (a.e.).

Corollary 97 (Reconcilation of the Fourier series to its Original Function under the Féjer Kernel)

Let $1 \leq p < \infty$. *If* $[f], [g] \in L_p(\mathbb{T}, \mathbb{C})$ *and* $\alpha_n^{[f]} = \alpha_n^{[g]}$ *for all* $n \in \mathbb{Z}$, *then* $[f] = [g]$.

Proof

Observe that $\alpha_n^{[f]} = \alpha_n^{[g]}$ for all $n \in \mathbb{Z}$, implies that $\sigma_N[f] = \sigma_N[g]$ for all $N \geq 1$. It follows from Corollary 96 that

$$[f] = \lim_{N \to \infty} \sigma_N[g] = \lim_{N \to \infty} \sigma_N[g] = [g].$$ □

22.2 *Which sequences are sequences of Fourier Coefficients?*

Given $[f] \in L_1(\mathbb{T}, \mathbb{C})$, we defined the Fourier series of $[f]$ as

$$\sum_{n \in \mathbb{Z}} \alpha_n^{[f]} [\xi_n].$$

The Riemann-Lebesgue Lemma stated that

$$(\alpha_n^{[f]})_{n \in \mathbb{Z}} \in c_0(\mathbb{Z}, \mathbb{C}).$$

It is then natural to ask if every sequence $(\beta_n)_{n \in \mathbb{Z}} \in c_0(\mathbb{Z}, \mathbb{C})$ is the sequence of coefficients of some $[f] \in L_1(\mathbb{T}, \mathbb{C})$. What we have seen is that on Hilbert spaces, every $(\gamma_n)_{n \in \mathbb{Z}} \in \ell_2(\mathbb{Z}, \mathbb{C})$ is the set of Fourier coefficients of some $[f] \in L_2(\mathbb{T}, \mathbb{C})$, and namely

$$[f] = \sum_{n \in \mathbb{Z}} \gamma_n [\xi_n].$$

We shall use **Operator Theory** to answer this. Recall that by the end of Section 17.1, we introduced the map

$$\begin{aligned} \Lambda : (L_1(\mathbb{T}, \mathbb{C}), \|\cdot\|_1) &\to (c_0(\mathbb{Z}, \mathbb{C}), \|\cdot\|_\infty) \\ [f] &\mapsto \left(\alpha_n^{[f]} \right)_{n \in \mathbb{Z}}. \end{aligned}$$

Since the Lebesgue integration is linear, so is Λ. Also, as shown before,

$$\left| \alpha_n^{[f]} \right| \leq \|[f]\|_1, \quad \forall n \in \mathbb{Z},$$

and so

$$\|\Lambda[f]\|_\infty = \sup\{ \left| \alpha_n^{[f]} \right| : n \in \mathbb{Z} \} \leq \|[f]\|_1.$$

Thus Λ is bounded, with $\|\Lambda\| \leq 1$.

By Corollary 97, if $[f], [g] \in L_1(\mathbb{T}, \mathbb{C})$ and $\Lambda[f] = \Lambda[g]$, then $[f] = [g]$, and thus Λ is injective.

Thus, our question of whether or not every sequence in $c_0(\mathbb{Z}, \mathbb{C})$ is a sequence of Fourier coefficients of some element of $L_1(\mathbb{T}, \mathbb{C})$ is therefore the question of whether or not Λ is surjective.

We require the **Inverse Mapping Theorem** from **Functional Analysis** to answer this question. To that end, we first introduce some notations.

Given a Banach space $(\mathfrak{Z}, \|\cdot\|_{\mathfrak{Z}})$ and a real number $r > 0$, we denote the **closed ball of radius r centered at the origin** by

$$\mathfrak{Z}_r = \{z \in \mathfrak{Z} : \|z\|_{\mathfrak{Z}} \leq r\}.$$

For $z_0 \in \mathfrak{Z}$ and $\varepsilon > 0$, we denote by $B^{\mathfrak{Z}}(z_0, \varepsilon) = \{z \in \mathfrak{Z} : \|z - z_0\| < \varepsilon\}$ the open ball of radius ε in $\mathfrak{Z}$, centered at z_0.

Lemma 98 (Finding an Open Container from a Closed Container)

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces and suppose that $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. If $\mathfrak{Y}_1 \subseteq \overline{T\mathfrak{X}_m}$ for some $m \geq 1$, then $\mathfrak{Y}_1 \subseteq T\mathfrak{X}_{2m}$.

Proof

To be added □

Theorem 99 (The Open Mapping Theorem)

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces and suppose that $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ is a surjection. Then T is an open map; i.e. if $G \subseteq \mathfrak{X}$ is open, then $TG \subseteq \mathfrak{Y}$ is open.

Proof

To be added □

Corollary 100 (The Inverse Mapping Theorem)

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces and suppose that $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ is a bijection. Then T^{-1} is continuous, and so T is a homeomorphism.

Proof

Since T is linear, by basic linear algebra, it has an inverse, which must also be linear.

If $G \subseteq \mathfrak{X}$ is open, then $(T^{-1})^{-1}(G) = TG$ is open in $\mathfrak{Y}$ by the Open Mapping Theorem. Thus T^{-1} is continuous, hence a homeomorphism. □

Theorem 101 ($L_1(\mathbb{T}, \mathbb{C})$ and $c_0(\mathbb{Z}, \mathbb{C})$ are Not Isomorphic)

The map

$$\begin{aligned} \Lambda : (L_1(\mathbb{T}, \mathbb{C}), \|\cdot\|_1) &\to (c_0(\mathbb{Z}, \mathbb{C}), \|\cdot\|_\infty) \\ [f] &\mapsto \left(\alpha_n^{[f]}\right)_{n\in\mathbb{Z}} \end{aligned}$$

is not surjective.

Proof

If it were surjective, then by the Inverse Mapping Theorem,

$$\begin{aligned} \Lambda^{-1} : c_0(\mathbb{Z}, \mathbb{C}) &\to L_1(\mathbb{T}, \mathbb{C}) \\ \left(\alpha_n^{[f]}\right)_{n\in\mathbb{Z}} &\mapsto [f] \end{aligned}$$

must be continuous.

Let D_N be the Dirichlet kernel of order N, and let $d_N := \Lambda[D_N]$, for $N \geq 1$. Then $d_N = (\ldots, 0, 0, \ldots, 0, 1, 1, \ldots, 1, 1, 0, 0, \ldots)$, where the 1's appear for the indices $-N \leq k \leq N$. It is clear that $\|d_N\|_\infty = 1$, since each d_N is **finitely supported**, [2] but by part 4 of

[2] This means that there are only finitely many non-zero values in its indices, which is the case.

Theorem 86,

$$\lim_{N\to\infty} \left\| \Lambda^{-1}(d_N) \right\|_1 = \lim_{N\to\infty} \|[D_N]\|_1 = \infty.$$

Thus Λ^{-1} is not continuous, a contradiction. Hence Λ must not be surjective. In other words, there exists a sequence $(\beta_n)_{n\in\mathbb{Z}} \in c_0(\mathbb{Z}, \mathbb{C})$ that are not Fourier coefficients of any element of $L_1(\mathbb{T}, \mathbb{C})$. □

Remark 22.2.1

As remarked, the fact that $[f] \in L_2(\mathbb{T}, \mathbb{C})$ iff $\left(\alpha_n^{[f]}\right)_{n\in\mathbb{Z}} \in \ell_2(\mathbb{Z}, \mathbb{C})$ makes it tempting to conjecture that perhaps the range of the map Λ from Theorem 101 should be $\ell_1(\mathbb{Z}, \mathbb{C})$, but that is not true at all, and it is not even surjective on $c_0(\mathbb{Z}, \mathbb{C}) \subseteq \ell_1(\mathbb{Z}, \mathbb{C})$.

For a clear example, the sequence

$$\beta_n = \begin{cases} \frac{1}{n} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

is clearly in $\ell_2(\mathbb{Z}, \mathbb{C})$, and so $[f] \coloneqq \sum_{n\in\mathbb{Z}} \beta_n [\xi_n]$ converges in $L_2(\mathbb{T}, \mathbb{C}) \subseteq L_1(\mathbb{T}, \mathbb{C})$. However,

$$\Lambda[f] = (\beta_n)_{n\in\mathbb{Z}}$$

is definitely not in $\ell_1(\mathbb{Z}, \mathbb{C})$.

A Interest in 2π periodic functions

This is ripped out of Professor Marcoux's [1] notes, which I think is rather important as a motivation to move from Lesbesgue's Theory of Integration into Fourier Analysis, but not important enough to warrant being added to the main section of the notes.

[1] Marcoux, L. W. (2019). *PMath 450 Introduction to Lebesgue Measure and Fourier Analysis*. (n.p.)

> *So where does the notation $L_1(\mathbb{T}, \mathbb{C})$ come from, given that we are dealing with 2π-periodic functions on $\mathbb{R}$? The issue lies in the fact that we are really interested in studying functions on $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, but that we have not yet defined what we mean by a measure on that set. We are therefore identifying $[-\pi, \pi)$ with $\mathbb{T}$ via the bijective function $\psi(\theta) = e^{i\theta}$. Thus, an alternative approach to this would be to say that a subset $E \subseteq T$ is measurable if and only if $\psi^{-1}(E) \subseteq [-\pi, \pi)$ is Lebesgue measurable. In order to "normalize" the measure of $\mathbb{T}$ (i.e. to make its measure equal to 1), we simply divide Lebesgue measure on $[-\pi, \pi)$ by 2π.*
> *This still doesn't quite explain why we are interested in 2π-periodic functions on $\mathbb{T}$, rather than just functions on $[-\pi, \pi)$, though. Here is the "kicker". The unit circle $\mathbb{T} \subseteq \mathbb{C}$ has a very special property, namely, that it is a group. Given $\theta_0 \in \mathbb{T}$, we can "rotate" a function $f : \mathbb{T} \to \mathbb{C}$ in the sense that we set $g(\theta) = f(\theta \cdot \theta_0)$. Observe that rotation along $\mathbb{T}$ corresponds to translation (modulo 2π) of the interval $[-\pi, \pi)$. The key is the irritating "modulo 2π" problem. If we don't use modular arithmetic, and if a function g is only defined on $[-\pi, \pi)$, we can not "translate" it, since the new function need no longer have $[-\pi, \pi)$ as its domain. We get around this by extending the domain of g to $\mathbb{R}$ and making g 2π-periodic. Then we may translate g by any real number $\tau_s^\circ(g)(\theta) := g(\theta - s)$, which has the effect*

that if we set $f(e^{i\theta}) = g(\theta)$, then $g(\theta - s) = f(e^{i\theta} \cdot e^{-is})$. That is, translation of g under addition corresponds to rotation of f under multiplication.

The last thing that we need to know is that such translations of functions will play a crucial role in our study of Fourier series of elements of $L_1(\mathbb{T}, \mathbb{C})$. Aside from being a Banach space, $L_1(\mathbb{T}, \mathbb{C})$ can be made into an algebra under convolution. While our analysis will not take us as far as that particular result, we will still need to delve into the theory of convolutions of continuous functions with functions in $L_1(\mathbb{T}, \mathbb{C})$. This will provide us with a way of understanding how and why various series associated to the Fourier series of an element $[f] \in L_1(\mathbb{T}, \mathbb{C})$ converge or diverge. Since convolutions are defined as averages under translation by the group action, and since $\mathbb{T}$ is a group under multiplication and $\mathbb{R}$ is a group under addition, our identification of $(\mathbb{T}, \cdot)$ with $([-\pi, \pi), +)$ (using modular arithmetic) is not an unreasonable way of doing things.

B Assignment Problems

B.1 Assignment 1 (A1)

Question 1 (Separated Sets)

Let A and B be bounded subsets of $\mathbb{R}$ and suppose that

$$\delta := \operatorname{dist}(A, B) := \inf\{|a - b| : a \in A,\ b \in B\} > 0.$$

Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.

Question 2 (A continuity result for Outer Measures)

Let $E \subseteq \mathbb{R}$. Prove that

$$\lim_{N \to \infty} m^*(E \cap [-N, N]) = m^*(E).$$

Question 3 (Finite Covers of $[0, 1]$)

Let $\Gamma = \mathbb{Q} \cap [0, 1]$. Prove that if $\{I_n\}_{n=1}^N$ is a finite collection of open intervals which covers Γ (i.e. $\Gamma \subseteq \bigcup_{n=1}^N I_n$), then $\sum_{n=1}^N \ell(I_n) \geq 1$.

Question 4 (Measures on Countable Sets)

*Let $X = \{x_n\}_{n=1}^\infty$ be a countable set, and recall that $\mathcal{P}(X)$ is the **power set** of X, i.e. $\mathcal{P}(X) = \{Y : Y \subseteq X\}$. A **measure** on X is a function*

$$\mu : \mathcal{P}(X) \to [0, \infty]$$

such that $\mu(\varnothing) = 0$ and for every disjoint sequence $E_n \subseteq X$, $n \geq 1$, we have

$$\mu\left(\bigcup_{n=1}^\infty E_n\right) = \sum_{n=1}^\infty \mu(E_n).$$

Let $\mathcal{M}(X) = \{\mu : \mu$ is a measure on $X\}$. Find a description of all possible measures on X; i.e., show that there exist sets A and B and a bijective map $\theta : \mathcal{M}(X) \to S := B^A$.

Question 5 (Open Subsets of $\mathbb{R}$)

Prove that if $G \subseteq \mathbb{R}$ is open, then G is a countable, disjoint union of open intervals.
Hint: *Define a relation on G via $x \sim y$ if $[\min\{x,y\}, \max\{x,y\}] \subseteq G$.*

Question 6 (Towards Borel Sets)

Let $E \subseteq \mathbb{R}$. We say that E is a G_δ-set if it is a countable intersection of open subsets of $\mathbb{R}$. We say that E is an F_σ-set if it is a countable union of closed subsets of $\mathbb{R}$. Recall that for $E \subseteq \mathbb{R}$, $E^C := \{x \in \mathbb{R} : x \notin E\}$ is the complement of E.

1. *Prove that every open set is an F_σ-set, and that every closed set is a G_δ-set.*
2. *Prove that the set of rational numbers is an F_σ-set, but not a G_δ-set.*
3. *Prove that the set of irrational numbers is a G_δ-set, but not an F_σ-set.*
4. *Let $E_1 = (-\infty, 0] \cap \mathbb{Q}^C$ and $E_2 = [0, \infty) \cap \mathbb{Q}$. Prove that $E := E_1 \cup E_2$ is neither a G_δ-set nor an F_σ-set.*

B.2 Assignment 2 (A2)

Question 1 (σ-additivity and continuity of the Lebesgue Measure)

1. *Let $\mathcal{L}(\mathbb{R})$ denote the set of Lebesgue measurable subsets of $\mathbb{R}$, and let $m : \mathcal{L}(\mathbb{R}) \to [0, \infty]$ denote the Lebesgue measure. Prove that m is σ-additive; i.e. if $E_n \in \mathcal{L}(\mathbb{R})$ for all $n \geq 1$ and $E_i \cap E_j = \emptyset$ if $1 \leq oi \neq j < \infty$, then*

$$m\left(\dot{\bigcup_{n=1}^{\infty}} E_n\right) = \sum_{n=1}^{\infty} m(E_n).$$

2. *Suppose that $\{E_n\}_{n=1}\infty$ is an increasing sequence of Lebesgue measurable sets; i.e.*

$$E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots.$$

Let $E = \bigcup_{n=1}^{\infty} E_n$, so that $E \in \mathcal{L}(\mathbb{R})$, as the latter is a σ-algebra. Prove that

$$mE = \lim_{n \to \infty} mE_n.$$

Question 2 (Continuity of the Lebesgue Measure II)

Let $\{E_n\}_{n=1}^{\infty}$ be a decreasing sequence of Lebesgue measurable sets; i.e.

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots.$$

Let $E = \bigcap_{n=1}^{\infty} E_n$.

1. *Suppose that $mE_1 < \infty$. Prove that*

$$mE = \lim_{n \to \infty} mE_n.$$

2. *Does the result of part 1 still hold if $mE_1 = \infty$? Prove that it does, or provide a counterexample to show that it need not be true.*

Question 3 (Lebesgue Inner Measure)

Let $E \subseteq \mathbb{R}$ be a set. Prove that the following are equivalent:

1. *E is measurable; i.e. $E \in \mathcal{L}(\mathbb{R})$;*
2. *For all $\varepsilon > 0$, there exists a closed set $F \subseteq E$ so that $m^*(E \setminus F) < \varepsilon$;*
3. *There exists an F_σ-set $H \subseteq E$ so that $m^*(E \setminus H) = 0$.*

Use this to show that if $E \subseteq \mathbb{R}$ is measurable, then

$$mE = \sup\{mK : K \subseteq E,\ K \text{ is compact }\}.$$

We say that the Lebesgue measure is ***regular****.*

Question 4 (σ-algebra of Sets)

1. *Let $\mathcal{A}$ be a σ-algebra of subsets of $\mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $\mathcal{B} = \{H \subseteq \mathbb{R} : f^{-1}(H) \in \mathcal{A}\}$. Show that $\mathcal{B}$ is a σ-algebra.*
2. *Recall that by definition, $f : \mathbb{R} \to \mathbb{R}$ is measurable if $f^{-1}(G) \subseteq \mathcal{L}(\mathbb{R})$ for all open sets $G \subseteq \mathbb{R}$. Use this definition to prove that f is measurable if and only if $f^{-1}(B) \subseteq \mathcal{L}(\mathbb{R})$ for all Borel sets $B \in \mathfrak{Bor}(\mathbb{R})$.*
3. *We say that $f : \mathbb{R} \to \mathbb{R}$ is* ***Borel measurable*** *if $f^{-1}((a, \infty)) \in \mathfrak{Bor}(\mathbb{R})$ for all $a \in \mathbb{R}$. Prove that f is Borel measurable if and only if $f^{-1}(B) \subseteq \mathfrak{Bor}(\mathbb{R})$ for all Bore sets $B \in \mathfrak{Bor}(\mathbb{R})$.*

Question 5 (The Cantor-Lebesgue Function)

Recall that we define the ***Cantor (middle third) set*** *C as $C = \bigcap_{n \geq 1} C_n$, where $C_0 = [0, 1]$, and for each $n \geq 1$,*

$$C_n = C_{n-1} \setminus \{I_{n,1} \cup I_{n,2} \cup \ldots \cup I_{n,2^{n-1}}\},$$

where $I_{n,j}$ is the open "middle third" of the j^{th} (closed) interval of C_{n-1}. If we set

$$G = \bigcup_{n \geq 1} \bigcup_{1 \leq j \leq 2^{n-1}} I_{n,j},$$

then the Cantor set is equal to $[0, 1] \setminus G$.

We define the ***Cantor-Lebesgue function*** *Γ_C on $[0, 1]$ as follows. For $x \in I_{n,j}$, we set $\Gamma_C(x) = \frac{2j-1}{2^n}$, $1 \leq j \leq 2^{n-1}$. We then extend Γ_C to all of $[0, 1]$ by setting $\Gamma_C(0) = 0$, and for $x \in (0, 1]$, we set*

$$\Gamma_C(x) = \sup\{\Gamma_C(t) : t \in [0, x) \cap G\}.$$

1. *Prove that the Cantor-Lebesgue function Γ_C is an increasing, continuous function that maps $[0, 1]$ onto $[0, 1]$.*
2. *Prove that if $\gamma(x) = \Gamma_C(x) + x$ for all $x \in [0, 1]$, then φ is a continu-*

ous function that maps $[0, 1]$ *onto* $[0, 2]$.

3. *Prove that* $\varphi(X) := \{\varphi(x) : x \in C\} \subseteq [0, 2]$ *is a measurable set of positive measure.*

Definition B.1 (Limit Superior and Limit Inferior)

Suppose that $(x_n)_n \in \mathbb{R}^{\mathbb{N}}$ *is a bounded sequence of real numbers. We define the* ***limit superior*** *(or* ***limit supremum****) of the sequence* $(x_n)_n$ *to be*

$$\limsup_{n \geq 1} x_n := \lim_{n \to \infty} \sup_{k \geq n} x_k,$$

and the ***limit inferior*** *(or* ***limit infimum****) to be*

$$\liminf_{n \geq 1} x_n := \lim_{n \to \infty} \inf_{k \geq n} x_k.$$

Setting $z_n : \sup_{k \geq n} x_k$*, for* $n \geq 1$*, we find that* $z_n \geq z_{n+1}$ *for all* $n \geq 1$*, and, from this, one should be able to convince themselves that* $\limsup_{n \geq 1} x_n$ *always exists, and similarly that* $\liminf_{n \geq 1} x_n$ *always exists. Moreover, if* $\mu \leq x_n \leq \nu$ *for all* $n \geq 1$ *(since we assumed that* $(x_n)_n$ *is bounded), then*

$$\mu \leq \liminf_{n \geq 1} x_n \leq \limsup_{n \geq 1} x_n \leq \nu.$$

If $(x_n)_n$ *is not bounded above, we define* $\limsup_{n \geq 1} x_n = \infty$*, while if* $(x_n)_n$ *is not bounded below, we define* $\liminf_{n \geq 1} x_n = -\infty$.

Question 6 (Lim Sups and Lim Infs — I)

Let $(x_n)_n \in \mathbb{R}^{\mathbb{N}}$ *be a bounded sequence of real numbers.*

1. *Prove that*

$$\limsup_{n \geq 1} x_n = \inf\{\gamma \in \mathbb{R} : \exists N > 0 \;\; \forall n \geq N \;\; x_n < \Gamma\}.$$

2. *Prove that if* $\limsup_{n \geq 1} x_n < \mu$*, then there exists* $N \in \mathbb{N}$ *such that* $n \geq N$ *implies that* $x_n < \mu$*. (We say that* $x_n < \mu$ *for large n.)*

3. *Prove that* $\limsup_{n\geq 1} x_n > \mu$ *implies that* $x_n > \mu$ *for infinitely many values of* $n \in \mathbb{N}$.

4. *Show that if* $(a_n)_n$ *and* $(b_n)_n$ *are bounded sequences of real numbers, then*

$$\limsup_{n\geq 1}(a_n + b_n) \leq \limsup_{n\geq 1} a_n + \limsup_{n\geq 1} b_n.$$

Give an example to show that equality need not occur.

Question 7 (Lim Sups and Lim Infs — II)

1. *Let* $(x_n)_n \in \mathbb{R}^{\mathbb{N}}$ *be a bounded sequence. Prove that if* $\beta := \limsup_{n\geq 1} x_n$, *then*

   (a) *there exist a subsequence* $(x_{n_k})_k$ *of* $(x_n)_n$ *such that* $lim_{k\to\infty} x_{n_k} = \beta$; *and*

   (b) *if* $(x_{m_k})_k$ *is any subsequence of* $(x_n)_n$ *which converges, say to some* $\alpha \in \mathbb{R}$, *then* $\alpha \leq \beta$.

   In other words, β *is the largest limit point of any subsequence of* $(x_n)_n$.

2. *Let* $(y_n)_n \in \mathbb{R}^{\mathbb{N}}$ *be a sequence. Prove that the following conditions are equivalent:*

   (a) *there exists* $\gamma \in \mathbb{R}$ *such that* $\lim_{n\to\infty} y_n = \gamma \in \mathbb{R}$; *i.e.* $(y_n)_n$ *is convergent (to* γ*); and*

   (b) $\limsup_{n\geq 1} y_n = \gamma = \liminf_{n\geq 1} y_n$.

B.3 Assignment 3 (A3)

Question 1 (Measurability of Extended Real-Valued Functions)

*Recall that $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ denotes the set of **extended real numbers**. Let $f : \mathbb{R} -> \overline{\mathbb{R}}$ be a function, and recall that f is said to be measurable if $f^{-1}(G) \in \mathcal{L}(\mathbb{R})$ for all open sets $G \subseteq \mathbb{R}$ and $f^{-1}(\{-\infty\}), f^{-1}(\{\infty\}) \in \mathcal{L}(\mathbb{R})$.*

Prove that the following are equivalent:

1. *f is measurable.*
2. *For all $\alpha \in \mathbb{R}$, $f^{-1}((\alpha, \infty]) \in \mathcal{L}(\mathbb{R})$.*
3. *For all $\beta \in \mathbb{R}$, $f^{-1}([-\infty \beta)) \in \mathcal{L}(\mathbb{R})$.*

Question 2 (Measurable Functions as Limits of Simple Functions)

1. *Let $f : \mathbb{R} \to [0, \infty]$ be a measurable function. Show that there exists an increasing sequence of measurable, simple functions $\varphi_n : \mathbb{R} \to [0, \infty)$ so that*
$$f(x) = \lim_{n \to \infty} \varphi_n(x), \quad \forall x \in \mathbb{R}.$$
2. *Let $E \in \mathcal{L}(\mathbb{R})$ and let $g : E \to [0, \infty]$ be a measurable function. Show that there exists an increasing sequence of measurable, simple functions $\psi_n : E \to [0, \infty)$ so that*
$$g(x) = \lim_{n \to \infty} \psi_n(x), \quad \forall x \in E.$$

***Hint for 1.**: For each $n \geq 1$, partition the interval $[0, n)$ into $n2^n$ equal subintervals $E_{k,n} = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$, $0 \leq k < (n2^n) - 1$. Let $E_{n2^n, n]} = [n, \infty]$. Use the sets $f^{-1}(E_{k,n})$, $0 \leq k \leq n2^n$ to build φ_n.*

***Hint for 2.**: This should be very short. Otherwise, you're doing something wrong.*

Question 3 (An Example)

1. *Let $E = [0, 1]$. Fix $m \geq 1$ and let $f : E \to \mathbb{R}$ be the function $f(x) = x^m$. Since f is continuous, f is measurable. Prove that the*

Lebesgue integral of f over E satisfies

$$\int_{[0,1]} f = \frac{1}{m+1}.$$

Note: You may not use ☕Theorem 40. You must prove this using only techniques available to Lebesgue integration, and it should suffice to do so with knowledge from before ☕Theorem 40.
*(**Hint**: The Monotone Convergence Theorem with the last problem should be useful.)*

2. *Let $g : E \to [0,1]$ be the function $g(x) = e^x$, i.e. the exponential function. Prove that the Lebesgue integral of g satisfies*

$$\int_0^1 g = e^1 - 1 = e - 1.$$

***Hint**: this should be much easier than part 1.*

Question 4 (Sets of Positive Measure are "Large")

1. *Let $E \in \mathfrak{M}(\mathbb{R})$ be a measurable set and suppose that $m(E) > 0$, i.e. E has strictly positive Lebesgue measure. Prove that the set*

$$H = E - E := \{x - y : x, y \in E\}$$

contains an interval.

Hints:

- *First, reduce to the case where $0 < mE < \infty$.*
- *Show that there exists an open interval $I = (a, b)$ so that $m(E \cap (a, b)) > 0.9(b - a)$.*
- *Let $F := E \cap (a, b) \subseteq E$. Suppose that $\alpha \in (-\delta, \delta)$, where $\delta = 0.1(b - a)$. Show that $F \cap (F + \alpha) \neq \emptyset$ by considering upper and lower bounds for the measure of $F \cup (F + \alpha)$.*

2. *Conclude that if $E \in \mathfrak{M}(\mathbb{R})$ satisfies $m(E) > 0$, then the cardinality of E coincides with $\mathfrak{c}$, the cardinality of $\mathbb{R}$.*

***Hint**: Part 2 is doable even if you did not get part 1.*

Question 5 (The Ubiquity of Non-Measurable Sets)

Prove that every measurable set E of (strictly) positive measure contains a non-measurable subset Z.

***Hints**: You may wish to reduce to the case where $E \subseteq [-N, N]$ for some $N > 0$. Then, you may wish to refer to the existence of non-measurable sets as demonstrated in the notes. You may DIY as well if you have an idea on what to do.*

Question 6 (Pointwise Convergence of Measurable Functions)

Let $E \in \mathfrak{M}(\mathbb{R})$ be a measurable set and suppose that $m(E) < \infty$. Let $(f_n)_n \in \mathcal{L}(E, \mathbb{R})$ and suppose that $(f_n)_n$ converges pointwise to a real-valued function $f : E \to \mathbb{R}$. Prove that if $\varepsilon > 0$ and $\delta > 0$, then there exists a measurable set $H \subseteq E$ and an integer $N \geq 1$ such that

1. *$m(H) < \delta$; and*
2. *$x \notin H$ implies that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$.*

B.4 Assignment 4 (A4)

Question 1 (Completeness in Normed Linear Spaces)

Let $(\mathfrak{X}, \|\cdot\|)$ *be a normed linear space. Prove that* $\mathfrak{X}$ *is complete, and hence a Banach space, if and only if every absolutely summable series in* $\mathfrak{X}$ *is summable. Here, a series* $\sum_{n=1}^{\infty} x_n$ *in* $\mathfrak{X}$ *is said to be* ***summable*** *if*

$$x := \lim_{N\to\infty} \sum_{n=1}^{N} x_n$$

exists in $\mathfrak{X}$*, while the series is said to be* ***absolutely summable*** *if*

$$\lim_{N\to\infty} \sum_{n=1}^{N} \|x_n\| < \infty.$$

Question 2 (ℓ_p-Spaces, Part I)

Let $1 \leq p < \infty$*. For* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$*, we define the* ℓ_p*-spaces:*

$$\ell_p = \ell_p(\mathbb{N}) := \left\{ x \in (x_n)_{n=1}^{\infty} \in \mathbb{K}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$

Furthermore, for $x \in \ell_p$*, we define* $\|x\|_p = \left(\sum_{n\geq 1} |x_n|^p\right)^{1/p}$*. When* $p = \infty$*, we define*

$$\ell_\infty = \ell_\infty(\mathbb{N}) := \left\{ x = (x_n)_{n=1}^{\infty} \in \mathbb{K}^{\mathbb{N}} \sup_{n\geq 1} |x_n| < \infty \right\}.$$

For $x \in \ell_\infty$*, we define* $\|x\|_\infty = \sup_{n\geq 1} |x_n|$*.*

1. *Prove Hólder's Inequality for* ℓ_p*-spaces: that is if* $1 \leq p \leq \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*, and if* $x = (x_n)_n \in \ell_p$ *and* $y = (y_n)_n \in \ell_q$*, then*

$$xy := (x_n y_n)_n \in \ell_1,$$

   and $\|xy\|_1 \leq \|x\|_p \|y\|_q$*.*

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