Final Exam Content Revision

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List of Theorems

Chapter 1

Important Theorems

Theorem 1.0.1 (Monotone Convergence Theorem)

Proof

Theorem 1.0.2 (Nested Interval Theorem)
For each $k \in \mathbb{N}$, if each I_k is closed, bounded, and nonempty. If $I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots$, then $\bigcap I_k \neq \emptyset$. Moreover, if $|I_k| \to 0$ as $k \to \infty$, $\bigcap I_k$ is a single point.

Proof

Theorem 1.0.3 (Intermediate Value Theorem)
Let $a, b \in \mathbb{R}$ with $a < b$, let $f : [a, b] \to \mathbb{R}$ be continous.

$$\forall y \in \mathbb{R} \;\; \min\{f(a), f(b)\} \leq y \leq \max\{f(a), f(b)\} \implies \exists x \in [a, b] \; f(x) = y$$

Proof

Theorem 1.0.4 (Closed Bounded Intervals and Uniform Convergence)
Let $a, b \in \mathbb{R}$ with $a < b$. If $f : [a, b] \to \mathbb{R}$ is continous, then f is uniformly continuous.

Theorem 1.0.5 (Equivalent Definitions of Integrability)
Let Let $a, b \in \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$. TFAE:

1. *f is integrable on $[a, b]$.*
2. *$\forall \epsilon > 0 \exists\ P$ that is a partition $U(f, P) - L(f, P) < \epsilon$*
3. *$U(f) = L(f)$*

Proof

Theorem 1.0.6 (Continuous Functions are Integrable)
Let $a, b \in \mathbb{R}$. Every continuous function $f : [a, b] \to \mathbb{R}$ is integrable.

Proof

Theorem 1.0.7 (Cauchy Criterion for Uniform Convergence - Sequence of Functions)
Let $I \subseteq \mathbb{R}$, $\forall n \in \mathbb{Z}^+\ f_n : I \to \mathbb{R}$.

$$\exists g : I \to \mathbb{R}\ f_n \to g \text{ uniformly on } I$$
$$\iff$$
$$\forall \epsilon > 0\ \exists N \in \mathbb{Z}^+\ \forall x \in I\ \forall k, l \in \mathbb{Z}^+\ (k, l \geq N \implies |f_k(x) - f_l(x)| < \epsilon)$$

Proof

Theorem 1.0.8 (Uniform Convergence, Limits and Continuity)
Supposed $f_n \to f$ uniformly on $I \subseteq \mathbb{R}$, where $\forall n \in \mathbb{Z}^+\ f_n : I \to \mathbb{R}$. If $\lim_{y \to x} f_n(y)$ exists for each n, then

$$\lim_{n\to\infty} \lim_{y\to x} f_n(y) = \lim_{y\to x} \lim_{n\to\infty} f_n(y)$$

Furthermore, if f_n is continuous on I for each n, then f is continous on I.

Proof

Theorem 1.0.9 (Equivalent Topological Definitions)
Let $A \subseteq \mathbb{R}^n$.

1. *A is closed iff $A' \subseteq A$*
2. *$\bar{A} = A' \cup A$*
3. *A^0 is equal to the set of all interior points of A*

4. $\partial A = \bar{A} \setminus A^0$

Proof

Theorem 1.0.10 (Heine-Borel Theorem)
Every subset of $\mathbb{R}^n$ *is compact iff it is closed and bounded.*

Proof

Chapter 2

Important Examples

Chapter 3

FAQ for Self

Chapter 4

Food for Thought