# ACTSC 431 — Loss Model I

Classnotes for Fall 2018

by

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## 1.1 Introduction and Overview

*Course Objective* In Loss Model I, the focus of our study is to learn the basic methods which are used by insurers to quantify risk from mathematical/statistical models, in order for insurers to make various decisions[1]. By quantifying risk, it helps us monitor underlying risks so that not only are we aware of them, but also so that we can take actions or preventive measures against them.

[1] e.g. setting premiums, control expenses, deciding for reinsurance, etc.

Our main interest of this course is:

- to quantify and seek protection against the loss of funds due either to **too many claims** or **a few large claims**;
- to reduce adverse financial impact of random events that prevent the realization of reasonable expectations.

The main model that shall be the focus of this course is **models for liability risk**.

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**Definition 1 (Liability Risk)**

*A* ***liability risk*** *is a risk that insurance companies assume by selling insurance contracts.*

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In particular, the liability that we shall focus on is **insurance claims**.

Many of the models that we shall see later in the course are also applied for other types of risks, e.g. investment risk, credit risk, liquidity risk, and operational risk.

We are interested in modelling the total amount of claims, i.e. the **aggregate claim amount**, of a group fo insurance policies over a given period of time. In the actuarial literature, there are two main approaches that have been proposed to model the aggrement claim amount of an insurance portfolio, namely:

- individual risk model;
- collective risk model.

### 1.1.1 Individual Risk Model

**Definition 2 (Individual Risk Model)**

*In an **individual risk model**, the aggregate claim is modeled by*

$$S = \sum_{i=1}^{n} Z_i$$

*where $n$ is a **deterministic**[2] integer that represents the **total number of insurance policies**, and $Z_i$ is a random variable for the **potential loss of the $i^{th}$ insurance policy.***

[2] i.e. fixed

**Note**

*Since a policy may or may not incur a loss[3], we have that*

$$P(Z_i = 0) > 0.$$

*Thus, in an individual risk model, we may also express the aggregate claim amount as*

$$S = \sum_{i=1}^{n} X_i I_i$$

*where $I_i$ is the indicator function about the claimant of policy $i$, while $X_i$ represents the size of the claim(s) for the $i^{th}$ policy provided that there is a claim.[4]*

[3] Since a claim may or may not be made!

[4] **This is actually incorrect, despite being in the recommended textbook. See Appendix E.1.**

However, in an individual risk model, according to Dhaene and Vyncke (2010)[5],

> *A third type of error that may arise when computing aggregate claims follows from the fact that the assumption of mutual independency of the individual claim amounts may be violated in practice.*

[5] Dhaene, J. and Vyncke, D. (2010). The individual risk model. https://www.researchgate.net/publication/228232062_The_Individual_Risk_Model

Due to complications such as this, the individual risk model will not be the focus of our studies.

### 1.1.2 Collective Risk Model

**Definition 3 (Collective Risk Model)**

*In a* ***collective risk model****, the aggregate claim is modeled by*

$$S = \sum_{i=1}^{N} X_i,$$

*where $N$ is a non-negative integer-valued random variable that denotes* ***the number of claims among a given set of policies****, while $X_i$ denotes the* ***size of the $i^{th}$ policy.***

**Note**

*In a collective risk model, we need to determine:*

- *the distribution of the total number of claims for the entire portfolio, i.e. the distribution of $N$; and*
- *the distribution of the loss amount per claim, i.e. the distribution of $X_i$.*

In this course, the primary focus of our studies will be on **collective risk models**.

*Terminologies* To end today's lecture, the following terminologies are introduced:

**Definition 4 (Severity Distribution)**

*The* ***severity distribution*** *is the distribution of the loss amount of the amount paid by the insurer on a given loss/claim.*

**Definition 5 (Frequency Distribution)**

*The **frequency distribution** is the distributino fo the number of losses/claims paid by the insurer over a given period of time.*

**Note**

*The frequency distribution is typically a discrete distribution.*

**Definition 6 (Aggrement Payment / Loss)**

*The **aggregate payment (loss)** is the total amout of all claim payments (losses) over a given period of time.*

**Note**

*There is a distinction between an aggregate payment and an aggregate loss, since an aggregate payment is "essentially" an aggregate loss after certain claim adjustments, such as deductibles, limits, and coinsurance.*

# 2 Lecture 2 Sep 11th

## 2.1 Review of Probability Theory

Firstly, we shall review the definition of a random variable.

**Definition 7 (Random Variable)**

*Let $\Omega$ be a sample space and $\mathcal{F}$ its $\sigma$-algebra[1]. A* ***random variable*** *(rv) $X : \Omega \to (\Omega, \mathcal{F})$ is a function from a possible set of outcomes to a measurable space $(\Omega, \mathcal{F})$. Within the context of our interest, $X$ is real-valued, i.e. $(\Omega, \mathcal{F}) = \mathbb{R}$.*

[1] For definitions of $\Omega$ and $\mathcal{F}$, see notes on STAT330.

### 2.1.1 Discrete Random Variables

**Definition 8 (Discrete Random Variable)**

*A* ***discrete random variable*** *(drv) is an rv $X$ that takes only countable (finite) real values.*

**Note**

*Let $X$ be a drv.*

- *The* ***probability mass function*** *(pmf) of $X$ is: for $i \in \mathbb{N}$,*

$$p(x_i) = P(X = x_i)$$

- *The **cumulative distribution function** (cdf) of X is*

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} p(x_i).$$

- *The kth **moment** of X is*[2]

$$E[X^k] = \sum_{i \in \mathbb{N}} x_i^k p(x_i)$$

*if $E[X^k]$ is finite.*

- *Some commonly seen/introduced discrete distributions are: Poisson, Binomial, Negative Binomial*

[2] This implicitly uses the Law of the Unconcious Statistician.

**Example 2.1.1**

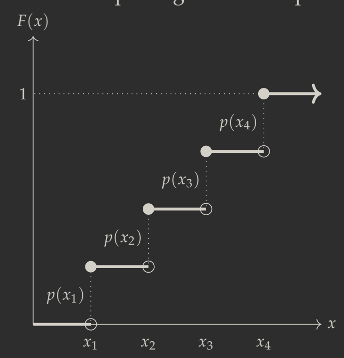
Let $X$ take values from $\{x_1, x_2, x_3, x_4\}$, and

$$p(x_i) = P(X = x_i) \text{ for } i = 1, 2, 3, 4.$$

The cdf of $X$ is

$$F(x) = \begin{cases} 0 & x < x_1 \\ p(x_1) & x_1 \leq x < x_2 \\ p(x_1) + p(x_2) & x_2 \leq x < x_3 \\ 1 - p(x_4) & x_3 \leq x < x_4 \\ 1 & x \geq x_4 \end{cases}$$

It is recommended to visualize the cdf first before putting it down in pencil.




**Note**

- *It is important that we stress the need for showing **right continuity** in the graph.*
- *Note that the cdf always sums to 1.*
- *The "**jumps**" at $x_i$ correspond to $p(x_i)$, for $i = 1, 2, 3, 4$.*

**Definition 9 (Probability Generating Function)**

*Suppose a drv X only takes **non-negative integer values**. The **proba-***

***bility generating function*** *(pgf) of $X$ is defined as*

$$G(z) = E\left[z^X\right] = \sum_{k=1}^{\infty} z^k p(k)$$

*where we note that if* $\max X = n$, *then* $p(m) = 0$ *for all* $m > n$.

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**Note**

- *The pgf uniquely identifies the distribution of the drv*[3].
- *To get the probability for* $k \in \{0, 1, 2, \ldots\}$, *we simply need to do*

$$p(k) = \frac{1}{k!} G^{(k)}(x)\Big|_{x=0}.$$

[3] This was given as is without proof, and I cannot find any resources that proves this.

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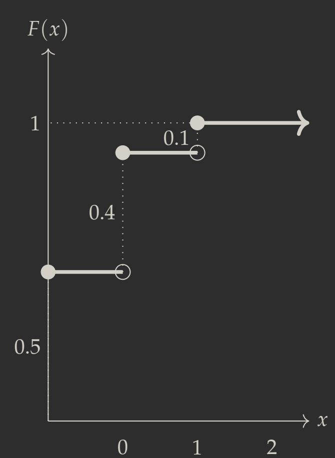
**Example 2.1.2 (Lecture Slides: Example 1)**

Consider a drv $X$ with pmf

$$p(x) = P(X = x) = \begin{cases} 0.5 & x = 0 \\ 0.4 & x = 1 \\ 0.1 & x = 2 \end{cases}$$

Its cdf is

$$F(x) = P(X \leq x) \begin{cases} 0 & x < 0 \\ 0.5 & 0 \leq x < 1 \\ 0.9 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$




and its pgf is

$$G(z) = E\left[z^X\right] = 0.5 + 0.4z + 0.1z^2.$$

### 2.1.2 *Continuous Random Variables*

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**Definition 10 (Continuous Random Variable)**

*A* ***continuous random variable*** *(crv) takes on a continuum of values.*

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**Note**

*Let $X$ be a crv.*

- *$\exists f : X \to \mathbb{R}$ called a* ***probability density function*** *(pdf) such that its cdf is*

$$F(x) = \int_{-\infty}^{x} f(y)\, dy,$$

*and consequently by the* ***Fundamental Theorem of Calculus****, we have*

$$f(x) = F'(x).$$

- *The kth moment of $X$ is*

$$E[X^k] = \int_x x^k f(x)\, dx$$

*so long that $E[X^k]$ is defined.*

- *Some commonly introduced distributions are: Uniform, Exponential, Gamma, Weibull, and Normal.*

**Definition 11 (Moment Generating Function)**

*Let $X$ be an rv. The* ***moment generating function*** *(mgf) of $X$ is, for $t \in \mathbb{R}$ (appropriately so),*

$$M_X(t) = E\left[e^{tX}\right] = \int_x e^{tx} f(x)\, dx$$

*provided that the integral is well-defined.*

The mgf is also defined for drvs.

**Note**

- *The mgf uniquely determines the distribution of its rv*[4]
- *With the mgf, we can obtain the kth moment of an rv $X$ by*

$$E\left[X^k\right] = \frac{d^k}{dt^k} M_X(t)\Big|_{t=0}$$

[4] ☝ This shall, also, not be proven in this course.

**Example 2.1.3 (Lecture Notes: Example 2)**

Consider an exponential rv $X$ with pdf[5]

[5] When not explicitly stated, it shall be assumed that domains at which we did not specify $x$ shall have probability 0.

$$f(x) = 0.1e^{-0.1x}, \; x > 0.$$

Its cdf is

$$F(x) = \int_{-\infty}^{x} f(y)\,dy = \begin{cases} 1 - e^{-0.1x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and its mgf is

$$\begin{aligned} M_X(t) = E\left[e^{tX}\right] &= \int_0^\infty e^{tx} 0.1 e^{-0.1x}\,dx \\ &= 0.1 \int_0^\infty e^{(t-0.1)x}\,dx \\ &= \frac{0.1}{0.1 - t}, \quad t < 0.1, \end{aligned}$$

where we note that we must have $t < 0.1$, for otherwise the value of the exponent would render the integral undefined.

### Definition 12 (Hazard Rate Function)

*For a crv $X$, the **hazard rate function** (aka **failure rate**) of $X$ is defined as*

$$h(x) = \frac{f(x)}{\overline{F}(x)} = -\frac{d}{dx} \ln \overline{F}(x),$$

*where $\overline{F}(x) = 1 - F(x)$ is the **survival function**[6]*

[6] You should be familiar with this if you have studied for Exam P.

### Note

- *We may also express the survival function in terms of the hazard rate by*

$$\overline{F}(x) = e^{-\int_{-\infty}^{x} h(y)\,dy}.$$

- *In terms of limits, we can express the hazard rate function, for small* *enough $\delta > 0$, as*

$$\begin{aligned} h(x) = \frac{f(x)}{\bar{F}(x)} = \frac{F'(x)}{\bar{F}(x)} \\ \approx \frac{F(x+\delta) - F(x)}{\delta \bar{F}(x)} \\ = \frac{P(x < X \leq x + \delta)}{\delta F(X > x)} \\ = \frac{1}{\delta} P(x < X \leq x + \delta \mid X > x). \end{aligned}$$

*We can make sense of this expression by recalling the notion of the probability of survival from Exam MLC*[7]*, where if a life has survived over $x$, the hazard rate is the probability that the life does not survive beyond another $\delta$* [8].

[7] This also tells us that the hazard rate gets its name from life insurance.

[8] From the perspective of life insurance, the greater the probability, the more likely the claim is going to happen.

# 3 Lecture 3 Sep 13th

## 3.1 Review of Probability Theory (Continued)

### 3.1.1 Continuous Random Variables (Continued)

**Example 3.1.1 (Lecture Notes: Example 3 — Hazard Rate of Weibull Distribution)**

Suppose $X \sim \mathrm{Wei}(\theta, \tau)$ with pdf

$$f(x) = \frac{\tau\left(\frac{x}{\theta}\right)^{\tau} e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{x}, \quad x > 0,$$

where $\theta, \tau > 0$. Find its hazard rate function.

**Solution**

We first require the survival function[1]:

[1] Weibull Survival Function

$$\begin{aligned}
\bar{F}(x) &= \int_x^\infty \frac{1}{y}\tau\left(\frac{y}{\theta}\right)^{\tau} e^{-\left(\frac{y}{\theta}\right)^{\tau}} \, dy \\
&= \int_{\frac{x}{\theta}}^\infty \frac{1}{u}\tau u^{\tau} e^{-u^{\tau}} \, du \qquad \text{where } u = \frac{y}{\theta} \\
&= \int_{\frac{x}{\theta}}^\infty \tau u^{\tau-1} e^{-u^{\tau}} \, du \\
&= -e^{-u^{\tau}} \Big|_{\frac{x}{\theta}}^{\infty} = e^{-\left(\frac{x}{\theta}\right)^{\tau}}
\end{aligned}$$

The hazard rate is therefore

$$h(x) = \frac{f(x)}{\bar{F}(x)} = \frac{\tau}{x}\left(\frac{x}{\theta}\right)^{\tau}$$

### 3.1.2 Mixed Random Variable

**Definition 13 (Mixed Random Variable)**

*We call $X$ a* ***mixed random variable*** *(mixed rv) if it has both discrete and continuous components.*

**Note**

- *Mixed rvs are important in modeling insurance claims, e.g., the loss amount is usually a continuous random variable with a probability mass at 0.*

The following is a type of mixed random variable:

**Definition 14 (Deductibles)**

*Let $X$ be an rv and $d$ be a fixed value.*

$$[X - d]_+ = \begin{cases} X - d & x \geq d \\ 0 & \text{otherwise} \end{cases}$$

**Note**

*If $X$ be an rv and $d$ a fixed value, the deductible $[X - d]_+$ has a mass point at 0 since*

$$P([X - d]_+ = 0) = P(X < d) > 0$$

**Note**

*Let $\{x_1, x_2, \ldots\}$ be a sequence of real numbers in an increasing order. Suppose $X$ is a rv that takes on values on the real, and has a* ***density function*** *$f$ on each interval $(x_i, x_{i+1})$, and has* ***discrete mass points*** *at the boundaries of these intervals, i.e.*

$$P(X = x_i) = p(x_i) > 0 \quad i \in \mathbb{N}.$$

In other words, we treat the discrete and continuous part of a mixed rv separately.

*Since $X$ is an rv, it must be the case that*

$$\sum_{i \in \mathbb{N}} p(x_i) + \sum_{i \in \mathbb{N}} \int_{x_i}^{x_{i+1}} f(x)\, dx = 1.$$

*The cdf of a mixed rv X is*

$$F(x) = P(X \leq x) = \sum_{i \in \mathbb{N}} p(x_i) \mathbb{1}_{\{x_i \leq x\}} + \sum_{i \in \mathbb{N}} \int_{x_i}^{x_{i+1}} f(y) \mathbb{1}_{\{y \leq x\}} \, dy.$$

*The kth moment of X is*

$$E\left[X^k\right] = \sum_{i \in \mathbb{N}} (x_i)^k p(x_i) + \sum_{i \in \mathbb{N}} \int_{x_i}^{x_{i+1}} x^k f(x) \, dx.$$

*The mgf of X is*

$$M_X(t) = E\left[e^{tX}\right] = \sum_{i \in \mathbb{N}} e^{tx_i} p(x_i) + \sum_{i \in \mathbb{N}} \int_{x_i}^{x_{i+1}} e^{tx} f(x) \, dx.$$

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**Example 3.1.2 (Lecture Notes: Example 4)**

Assume a claim amount of an insurance policy is modeled by a non-negative rv $X$ which has probability mass of $p$ and 0, and otherwise continuous with a pdf $f$ over $(0, \infty)$. Find its cdf, $k$th moment, and mgf.

**Solution**
The cdf of $X$ is

$$F(x) = \begin{cases} p + \int_0^x f(y) \, dy & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The $k$th moment of $X$ is

$$E\left[X^k\right] = \int_0^\infty x^k f(x) \, dx.$$

The mgf of $X$ is

$$M_X(t) = p + \int_0^\infty e^{tx} f(x) \, dx.$$

## 3.2 *Distributional Quantities and Risk Measures*

This chapter introduces us to some **distributional quantities** for a given rv $X$. These distributional quantities are informative values to describe the characteristics of a risk.

### 3.2.1 *Distributional Quantities*

**Definition 15 (Central Moment)**

*The **kth central moment** of an rv $X$ is defined as*

$$E\left[(X - E(X))^k\right].$$

**Note**

*The second central moment is the **variance**. The square root of the variance is the **standard deviation**.*

**Example 3.2.1 (Lecture Notes: Example 5)**

Consider an rv $Y = \begin{cases} Y_1 & U = 1 \\ Y_2 & U = 2 \end{cases}$ [2], where $Y_1 = 0$, $Y_2 \sim \text{Exp}(10)$, and $P(U = 1) = P(U = 2) = 0.5$.

[2] This notation is just syntatic sugar for saying $Y_1 = Y \mid (U = 1)$ and $Y_2 = Y \mid (U = 2)$.

1. Find the cdf of $Y$.
2. Find the mean and variance of $Y$.
3. Let $Z = \frac{1}{2}Y_1 + \frac{1}{2}Y_2$. Does $Z$ have the same distribution as $Y$? Answer this by solving the mean and variance of $Z$.

**Solution**

1. Note that

$$F(y) = P(Y_1 \leq y \mid U = 1)P(U = 1) + P(Y_2 \leq y \mid U = 2)P(U = 2).$$

Observe that

$$P(Y_1 \leq y \mid U = 1) = \begin{cases} 1 & y \geq 0 \\ 0 & y < 0 \end{cases}$$

and

$$P(Y_2 \leq y \mid U = 2) = \begin{cases} 1 - e^{-10y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

Therefore

$$F(y) = \begin{cases} 1 - \frac{1}{2}e^{-10y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

2. The mean of $Y$ is

$$E(Y) = E(Y \mid U = 1)P(U = 1) + E(Y \mid U = 2)P(U = 2) = 10 \cdot \frac{1}{2} = 5.$$

To calculate the variance of $Y$, we require

$$\begin{aligned} E\left[Y^2\right] &= E\left[Y^2 \mid U = 1\right] P(U = 1) + E\left[Y^2 \mid U = 2\right] P(U = 2) \\ &= (\mathrm{Var}(Y_2) + E(Y_2)^2) \cdot \frac{1}{2} = 100. \end{aligned}$$

Therefore

$$\mathrm{Var}(Y) = 100 - 5^2 = 75.$$

3. The mean of $Z$ is

$$E[Z] = E[\frac{1}{2}Y_1 + \frac{1}{2}Y_2] = 5.$$

The variance of $Z$ is

$$\mathrm{Var}(Z) = \frac{1}{4}\mathrm{Var}(Y_1) + \frac{1}{4}\mathrm{Var}(Y_2) = 25.$$

Therefore, $Z$ does not have the same distribution as $Y$.

**Definition 16 (Quantiles)**

*The $100p\%$ **quantile** (or **percentile**) of an rv $X$ is a set $\pi_p$ such that*

$$\pi_p = \{x \in X \mid P(X < x) \leq p \leq P(X \leq x)\}.$$

This definition may also be presented as: any number $\pi_p$ such that

$$P(X < \pi_p) \leq p \leq P(X \leq \pi_p).$$

**Note**

- *If $X$ is a continuous random variable, we have that $P(X < \pi_p) = P(X \leq \pi_p)$ and so we have to define the quantile as*

  $$\pi_p = F^{-1}(p)$$

  *where $F^{-1}$ is the inverse function of $F$, the cdf of $X$.*

- *A quantile **can be a set of numbers**.*

- *$\pi_{0.5}$ is called the **median** of $X$.*

Graphical method to interpret this notion will be included.

**Example 3.2.2 (Lecture Notes: Example 1)**

Find the $100p\%$ quantile of the loss distribution $F(x) = 1 - e^{-\frac{x}{\theta}}$, $x > 0$.

**Solution**

Note that $F$ is the cdf of an exponential distribution, which is a continuous distribution. Therefore,

$$F(\pi_p) = 1 - e^{-\frac{\pi_p}{\theta}} = p \implies \pi_p = -\theta \ln(1 - p).$$

**Example 3.2.3 (Lecture Notes: Example 2)**

Find the median $\pi_{0.5}$ for the following cdf

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.6 + 0.4(1 - e^{-\frac{x}{3}}) & x \geq 0 \end{cases}$$

**Solution**

Since $F(0) = 0.6$ and $F$ is an increasing function, we have that $F(x) = 0$ for all $x < 0$. Therefore

$$\pi_{0.5} = 0.$$

**Example 3.2.4 (Lecture Notes: Example 3)**

Find the median $\pi_{0.5}$ for a loss $X$ with pmf

$$p(0) = 0.25,\ p(1) = 0.25,\ p(2) = 0.5.$$

**Solution**

The cdf of $X$ is

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.25 & 0 \leq x < 1 \\ 0.5 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

since $F(x) = 0.5$ when $1 \leq x < 2$, we have that

$$\pi_{0.5} = [1, 2].$$

# 4 Lecture 4 Sep 18th

## 4.1 Distributional Quantities and Risk Measures (Continued)

### 4.1.1 Risk Measures

**Definition 17 (Risk Measure)**

*A **risk measure** is a mapping from the loss rv to the real line $\mathbb{R}$.*

Klugman, Panjer & Wilmot (2012) [1] on risk measure:

> *The level of exposure to risk is often described by one number, or at least a small set of numbers. These numbers are necessarily functions of the model and are often called 'key risk indicators'. Such key risk indicators indicate to risk managers the degree to which the company is subject to particular aspects of risk.*

[1] Klugman, S. A., Panjer, H. H., and Willmot, G. E. (2012). *Loss Models: From Data to Decisions*. John Wiley & Sons, Inc., 4th edition

To ensure its solvency, insurers will have to charge on these risks, i.e. we have to **price these exposures to risks**.

**Definition 18 (Premium Principle)**

*A **premium principle** (or **insurance pricing**) is a rule for assigning a premium to an insurance risk.*

**Note**

*The following are some of the common principles used by insurers:*

- ***Expectation Principle***

$$\Pi(X) = (1+\theta)E(X), \quad \theta > 0$$

- ***Standard Deviation Principle***

$$\Pi(X) = E(X) + \theta\sqrt{\mathrm{Var}(X)}, \quad \theta > 0$$

- ***Dutch Principle***

$$\Pi(X) = E(X) + \theta E([X - E(X)]_+), \quad \theta > 0$$

One particular measure is known as the **Value-at-Risk** (VaR).

#### 4.1.1.1 *Value-At-Risk*

**Definition 19 (Value-at-Risk (VaR))**

*The **Value-at-Risk (VaR)** is a **quantile** of the distribution of aggregate losses, i.e. the VaR of a risk X at the 100%p level is defined as*[2]

$$\begin{aligned}\mathrm{VaR}_p(X) &= \inf\{x \in \mathbb{R} : P(X > x) \leq 1 - p\} \\ &= \inf\{x \in \mathbb{R} : P(X \leq x) \geq p\}.\end{aligned}$$

[2] I must find out why we define using inf instead of min (see following remark), and I will not take "safe definition" as an answer without full justification.

**Note**

- VaR *is often called a* ***quantile risk measure****.*
- VaR *is the standard risk measure used to evaluate exposure to risks.*
- VaR *measures the amount of capital required by the insurer to remain solvent, with high certainty, in the face of large claims.*
- *In practice, p is generally high:* 99.95% *or as low as* 95%*.*

**Remark**

*Observe that*

$$B = \{x \in \mathbb{R} \mid F_X(x) \geq p\} = (A, \infty) \text{ or } [A, \infty)$$

This remark basically points out that the left endpoint of the interval $B$ is always included, which should be quite clear by right-continuity of $F$.

*for some $A \in \mathbb{R}$, since $F$ is an increasing function. Now let $x_0 \in B$ such that*

$$F(x_0) = P(X \leq x_0) \geq p \quad \wedge \quad F(x_0-) = P(X < x_0) \leq p,$$

*i.e. it is not necessary that $P(X = x_0) = p$ (see the two example graphs on the margin).*

*Let $\{x_n\}_{n \in \mathbb{N}}$ be a decreasing sequence of points on $\mathbb{R}$ such that $x_n \to x_0$ as $n \to \infty$. Since $F$ is right-continuous, we have that $F(x_n) \to F(x_0)$ as $n \to \infty$. Therefore,*

$$B = [x_0, \infty)$$

*This justifies the definition of $\pi_p$.*

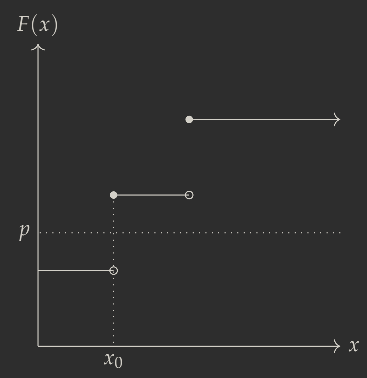


Figure 4.1: Discrete cdf

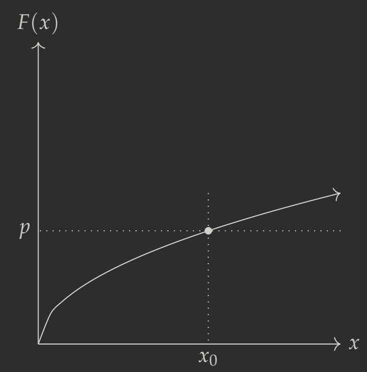


Figure 4.2: Continuous cdf

The lecturer asserts that we can really define VaR using min instead of inf, but even with this, I am not completely satisfied or convinced.

**Note**

- *Note that by definition, we have*

$$P(X < \pi_p) \leq p \leq P(X \leq \pi_p)$$

- *If $X$ is a crv whose cdf is strictly increasing, i.e. no constant points, then*

$$\pi_p = F^{-1}(p)$$

  *since $P(X < \pi_p) = P(X \leq \pi_p)$.*

**Warning (Shortcomings of VaR)**

- VaR *cannot tell us the size of the potential loss in the $100(1-p)\%$ cases, making it difficult for us to prepare the right amount in order to safeguard against insolvency.*
- VaR *actually fails to satisfy properties to be a* ***coherent risk measure***[3]*, for example,* ***subadditivity***.
- VaR *is extensively used in financial risk management of trading risk over a fixed (usually short) time period, which are usually normally distributed, and* VaR *satisfies all coherency requirements.*
- *In insurance losses, instead of normal distributions, in general, skewed distributions are used, and in this cases,* VaR *is flawed as it lacks subadditivity.*

[3] See Appendix E.2.

**Example 4.1.1**

Suppose that $X$ has a Pareto distribution with cdf

$$F(x) = 1 - \left( \frac{\theta}{x+\theta} \right)^{\alpha}, \quad x > 0$$

where $\alpha, \theta > 0$. Find $\mathrm{VaR}_p(X)$.

**Solution**

Since $F$ is continuous and strictly increasing, we have that

$$\pi_p = F^{-1}(p) = \theta \left[ (1-p)^{-\frac{1}{\alpha}} - 1 \right]$$

**Example 4.1.2**

Find $\mathrm{VaR}_{0.95}(X)$, $\mathrm{VaR}_{0.5}(X)$, and $\mathrm{VaR}_{0.3}(X)$ for a random loss with pmf

$$p(0) = 0.25, \; p(1) = 0.25, \text{ and } p(2) = 0.5.$$

**Solution**

Note that the cdf of $X$ is

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.25 & 0 \leq x < 1 \\ 0.5 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}.$$

Therefore,

$$\mathrm{VaR}_{0.95}(X) = 2, \; \mathrm{VaR}_{0.5}(X) = 1, \text{ and } \mathrm{VaR}_{0.3}(X) = 1.$$

### 4.1.1.2 *Tail-Value-at-Risk*

To compensate for the weakness of VaR at giving us the size of the loss $X$ of which we cannot measure, we use the **Tail-Value-at-Risk**.

**Definition 20 (Tail-Value-at-Risk (TVaR))**

*Let $X$ be an rv. The **Tail-Value-at-Risk (TVaR)** of $X$ at the $100p\%$ level, denoted as $\mathrm{TVaR}_p(X)$, is defined as the average of all VaR values above the level $p$, and expressed as*

TVaR also has the following names, used by different regions:

- **Conditional Tail Expectation** (CTE) — NA
- **Tail Conditional Expectation** (TCE)
- **Expected Shortfall** (ES) — EU

$$\text{TVaR}_p(X) = \frac{1}{1-p} \int_p^1 \text{VaR}_\alpha(X)\, d\alpha = \frac{1}{1-p} \int_p^1 \pi_\alpha \, d\alpha$$

**Remark**
*By considering the average of* VaR *from* $p$*'s going up to* $1$*, we take into account even the extreme cases of which* VaR *fails to account for.*

Perhaps a clearer definition would be the following, although the expression is only sensible if $X$ is a crv:

### Definition 21 (Tail-Value-at-Risk (TVaR))

*Let* $X$ *be an rv. The* ***Tail-Value-at-Risk (TVAR)*** *of* $X$ *at the* $100p\%$ *level, denoted* $\text{TVaR}_p(X)$*, is the expected loss given that the loss exceeds the* $100p$ *percentile (or quantile) of the distribution of* $X$*, expressible as*

$$\text{TVaR}_p(X) = E[X \mid X > \pi_p] = \frac{1}{\bar{F}(\pi_p)} \int_{\pi_p}^{\infty} x f(x)\, dx.$$

Note that the two definitions agree with one another:

$$\begin{aligned} \frac{1}{1-p} \int_p^1 \pi_\alpha \, d\alpha &= \frac{1}{1-F(\pi_p)} \int_p^1 F^{-1}(\alpha)\, d\alpha \\ &= \frac{1}{\bar{F}(\pi_p)} \int_{\pi_p}^1 x f(x)\, dx \end{aligned}$$

where we let $\alpha = F(x)$ as substitution.

### Note
*While it is not difficult to notice that*

$$\text{TVaR}_p(X) \geq \text{VaR}_p(X),$$

*the proof is also simple:*

$$\begin{aligned} \text{TVaR}_p(X) &= \frac{1}{1-p} \int_p^1 \pi_\alpha \, d\alpha \\ &\geq \frac{1}{1-p} \pi_p \int_p^1 d\alpha = \pi_p = \text{VaR}_p(X). \end{aligned}$$

**Example 4.1.3**

Find $\mathrm{TVaR}_p(X)$ for $X \sim \mathrm{Exp}(\theta)$.

**Solution**

Since $X$ is a crv, and $F(x) = 1 - e^{-\frac{x}{\theta}}$, we have that

$$\pi_p = F^{-1}(p) = -\theta \ln(1-p).$$

Therefore,

$$\begin{aligned}
\mathrm{TVaR}_p(X) &= \frac{1}{1-p}\int_p^1 \pi_\alpha \, d\alpha = \frac{-\theta}{1-p}\int_p^1 \ln(1-\alpha)\, d\alpha \\
&= \frac{-\theta}{1-p}\int_{-\infty}^{\ln(1-p)} ue^u \, du \quad \text{let } u = \ln(1-\alpha) \\
&= \frac{-\theta}{1-p}\left[ ue^u \Big|_{-\infty}^{\ln(1-p)} - \int_{-\infty}^{\ln(1-p)} e^u \, du \right] \text{ by IBP} \\
&= \frac{-\theta}{1-p}\left[(1-p)\ln(1-p) - (1-p)\right] \\
&= \theta[1 - \ln(1-p)]
\end{aligned}$$

**Note**

*From the last example, by the memoryless property of* $\mathrm{Exp}(\theta)$*, notice that we may also do*

$$\begin{aligned}
\mathrm{TVaR}_p(X) &= E[X \mid X > \pi_p] = E[X - \pi_p + \pi_p \mid X > \pi_p] \\
&= E[X - \pi_p \mid X > \pi_p] + E[\pi_p \mid X > \pi_p] \qquad (4.1) \\
&= E[X] + \pi_p
\end{aligned}$$

# 5 Lecture 5 Sep 20th

## 5.1 Distrbutional Quantities and Risk Measures (Continued 2)

### 5.1.1 Risk Measures (Continued)

Before ending this section, we introduce a notion that is related to TVaR.

**Definition 22 (Mean Excess Loss)**

*Let $X$ be an rv, and $d \in \mathbb{R}$. The **mean excess loss**, denoted $e_X(d)$, is defined as*

$$e_X(d) = E[X - d \mid X > d]$$

*and $e_X(d) = 0$ for those $d$ such that $P(X > d) = 0$.*

**Proposition 1 (Relation of $\mathrm{TVaR}_p(X)$ and $e_X(d)$)**

*For a crv $X$, we have*

$$\mathrm{TVaR}_p(X) = e_X(\pi_p) + \mathrm{VaR}_p(X)$$

**Proof**

By Equation (4.1), we have that

$$\mathrm{TVaR}_p(X) = E[X - \pi_p \mid X > \pi_p] + \pi_p = e_X(\pi_p) + \pi_p.$$

□

**Proposition 2 (Expection from Survival Function)**

*Let $X$ be a non-negative rv such that $E[X^k] < \infty$, for any $k \in \mathbb{N} \setminus \{0\}$. Then*[1]

$$E\left[X^k\right] = k \int_0^\infty x^{k-1} \bar{F}(x)\, dx$$

[1] Note that this works for the discrete case as well, by replacing $\int$ with $\sum$.

**Proof**

Firstly, note that since $E[X^k] < \infty$ for all $k \in \mathbb{N} \setminus \{0\}$, we have that $\bar{F}(x)$ decays faster than $x^k$ as $x \to \infty$. Now

$$\begin{aligned}
E\left[X^k\right] &= \int_0^\infty x^k f(x)\, dx \quad \because \text{ Law of the Unconscious Statistician} \\
&= \int_0^\infty x^k \, dF(x) \quad \because dF(x) = f(x)\, dx \\
&= -\int_0^\infty x^k \, d\bar{F}(x) \\
&= -\left[ x^k \bar{F}(x) \Big|_0^\infty - \int_0^\infty k x^{k-1} \bar{F}(x)\, dx \right] \quad \because \text{ IBP} \\
&= k \int_0^\infty x^{k-1} \bar{F}(x)\, dx
\end{aligned}$$

□

**Example 5.1.1**

Calculate $e_X(d)$ and $\mathrm{TVaR}_p(X)$ for a Pareto distribution $X$ with cdf

$$F(x) = 1 - \left( \frac{\theta}{x + \theta} \right)^\alpha, \quad x > 0,$$

where $\alpha > 1$ and $\theta > 0$.

**Solution**

Using Proposition 2,

$$\begin{aligned} e_X(d) &= \int_0^\infty P(X-d > x \mid X > d)\,dx = \int_0^\infty \frac{P(X-d>x, X>d)}{P(X>d)}\,dx \\ &= \int_0^\infty \frac{P(X>x+d)}{P(X>d)}\,dx = \int_0^\infty \frac{\bar{F}(x+d)}{\bar{F}(d)}\,dx \\ &= \int_0^\infty \left(\frac{d+\theta}{x+d+\theta}\right)^\alpha dx = \frac{(d+\theta)^\alpha}{1-\alpha}\left(\frac{1}{x+d+\theta}\right)^{\alpha-1}\bigg|_0^\infty \\ &= \frac{d+\theta}{\alpha-1} \end{aligned}$$

By Example 4.1.1, we have

$$\pi_p = \theta\left[(1-p)^{-\frac{1}{\alpha}} - 1\right]$$

and so

$$\begin{aligned} \mathrm{TVaR}_p(X) &= e_X(\pi_p) + \pi_p \\ &= \frac{\theta\left[(1-p)^{-\frac{1}{\alpha}}-1\right]+\theta}{\alpha-1} + \theta\left[(1-p)^{-\frac{1}{\alpha}}-1\right] \\ &= \frac{\theta(1-p)^{-\frac{1}{\alpha}}}{\alpha-1} + \frac{\theta(\alpha-1)(1-p)^{-\frac{1}{\alpha}}}{\alpha-1} - \theta \\ &= \frac{\theta\alpha(1-p)^{-\frac{1}{\alpha}}}{\alpha-1} - \theta \end{aligned}$$

**Proposition 3 (Expected Deductible)**

*We have*

$$E([X-d]_+) = \int_d^\infty \bar{F}(x)\,dx$$

**Proof**

By the Law of the Unconscious Statistician and IBP on the last step,

$$E([X-d]_+) = \int_d^\infty (x-d)\,dF(x) = -\int_d^\infty (x-d)\,d\bar{F}(x) = \int_d^\infty \bar{F}(x)\,dx$$

□

**Proposition 4 (An Expression for Mean Excess Value)**

*If $\bar{F}(d) > 0$, we have*

$$e_X(d) = \frac{\int_d^\infty \bar{F}(x)\,dx}{\bar{F}(d)}$$

**Proof**

Observe that by Proposition 3, we have

$$\begin{aligned} e_X(d) &= E[X - d \mid X > d] = \frac{E[(X-d)\mathbb{1}_{X>d}]}{P(X>d)} \\ &= \frac{E([X-d]_+)}{\bar{F}(d)} = \frac{\int_d^\infty \bar{F}(x)\,dx}{\bar{F}(d)} \end{aligned}$$

□

## 5.2 Severity Distributions — Creating Severity Distributions

Recall the definition of a severity distribution.

**Definition (Severity Distribution)**

*A* ***severity distribution*** *is a distribution used to describe single random losses in an insurance portfolio.*

When a loss occurs, the full amount of the loss is not necessarily the amount paid by the insurer, since an insurance policy typically involves some form of adjustment (e.g. **deductible, limit, coinsurance**). A distinction needs to be made between the actual loss prior to any of the adjustments (aka **ground-up loss**) and the amount ultimately paid by the insurer.

Our goal is to find a reasonable model for the **ground-up loss** rv $X$. The following are two desirable properties for $X$:

- $\mathrm{Im}(X) = \mathbb{R}_{>0}$, since losses are positive;
- pf of $X$ is right-skewed, since we want the "tail" of the distribution to be not heavy.

- The motivation for this property is due to the **20-80 rule**: 20% of the largest claims accountn for 80% of the total claim amount.

THERE ARE two approaches to constructing a severity distribution:

- **Parametric approach**[2]: specify a "form" for the distribution with a finite number of parameters.
- Nonparametric approach: no form is specified; the distribution is constructed directly from the empirical data.

[2] This approach shall be the focus of this course.

A weakness of the **Nonparametric approach** is, if there is not enough data, such as in catasthropic risks, is becomes difficult to obtain reliable information. We shall look at one such example in this approach.

**Definition 23 (Empirical Distribution Function)**

*Let $\{X_1, \ldots, X_n\}$ be an iid sample of a risk $X$. Then its **empricial distribution function (edf)** is defined as*

$$\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{X_i \leq x\}}, \quad x \in \mathbb{R}.$$

**Remark**

*Simply put, the edf assigns a probability of $\frac{1}{n}$ to each sample point $X_i$.*

**Example 5.2.1**

Consider a random sample of a risk with size 5: $\{30, 80, 150, 150, 200\}$. Find the edf of the risk.

**Solution**

The edf is given by

$$\hat{F}_n(x) = \frac{1}{5}\sum_{i=1}^{5} \mathbb{1}_{\{X_i \leq x_i\}} = \begin{cases} 0 & x < 30 \\ \frac{1}{5} & 30 \leq x < 80 \\ \frac{2}{5} & 80 \leq x < 150 \\ \frac{4}{5} & 150 \leq x < 200 \\ 1 & x \geq 200 \end{cases}$$

# 6 Lecture 6 Sep 25th

## 6.1 Severity Distributions — Creating Severity Distributions (Continued)

*The Parametric Approach* The following is a graph showing the process of a parametric approach:

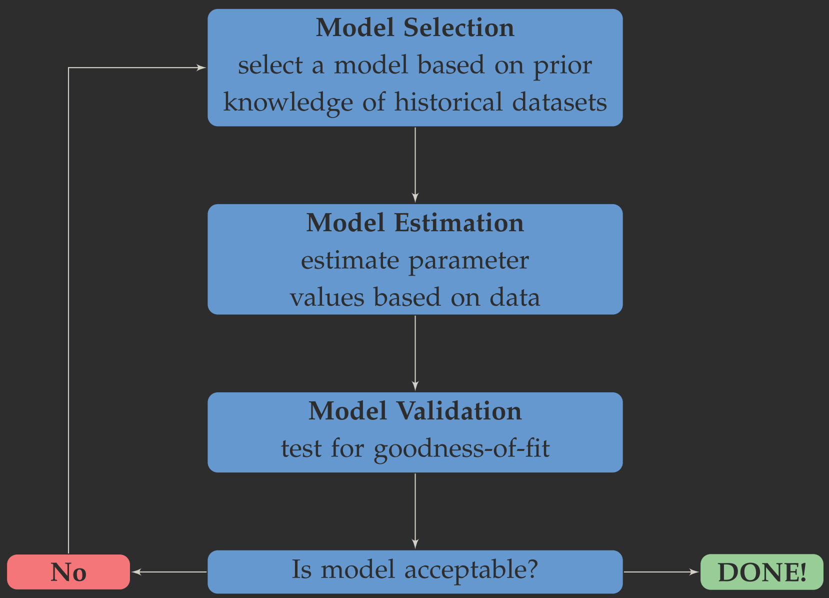


Figure 6.1: Process of a Parametric Approach

*Common Techniques in Creating New Parametric Distributions* Before diving into the topic, first, a definition:

**Definition 24 (Parametric Distribution)**

*A **parametric distribution** is a set of distribution functions, of which each member is determined by specifying one or more parameters.*

Some common techniques are the following:

- Multiplication by a constant
- Raising to a power
- Exponentiation
- Mixture of distributions

### 6.1.1 *Multiplication By A Constant*

This transformation is equivalent to applying inflation uniformly across all loss levels, and is known as a change of scale.

**Proposition 5 (Multiplication by a Constant)**

*Let $X$ be a crv with cdf $F_X$ and pdf $f_X$. Let $Y = cX$ for some $c > 0$. Then*

$$F_Y(y) = F_X\left(\frac{y}{c}\right), \quad f_Y(y) = \frac{1}{c} f_X\left(\frac{y}{c}\right).$$

**Proof**

$$F_Y(y) = P(Y \leq y) = P(cX \leq y) = P\left(X \leq \frac{y}{c}\right) = F_X\left(\frac{y}{c}\right)$$
$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X\left(\frac{y}{c}\right) = \frac{1}{c} f_X\left(\frac{y}{c}\right)$$

□

**Definition 25 (Scale Distribution)**

*We say that a parametric distribution is a* ***scale distribution*** *if $Y = cY$ for any positive constant $c$ is from the same set of distributions as $X$.*

It is clear that we have the following result:

**Corollary 6**

*The parameter $c$ in Proposition 5 is a scale parameter, and $Y$ is a scale distribution.*

**Example 6.1.1**

Let $X \sim \text{Exp}(\theta)$ with pdf

$$f_X(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x > 0.$$

Let $y = cX$ with $c > 0$, it follows that

$$f_Y(y) = \frac{1}{c} f_X\left(\frac{y}{c}\right) = \frac{1}{c\theta} e^{-\frac{y}{c\theta}}, \quad y > 0.$$

Thus $Y \sim \text{Exp}(c\theta)$ and so $Y$ is a scale distribution. In particular, the exponential distribution belongs to a family of scale distributions.

**Definition 26 (Scale Parameter)**

*A parameter $\theta$ is called a* ***scale paramter*** *of a parametric distribution $X$ if it satisfies the following condition: the parametric value of $cX$ is $c\theta$ for any positive constant $c$, and other parameters (if any) remain unchanged.*

**Example 6.1.2**

From Example 6.1.1, we had that

$$f_X(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x > 0.$$

We showed that $Y = cX \sim \text{Exp}(c\theta)$. Therefore, the parameter $\theta$ is a scale parameter.

**Example 6.1.3**

Determine whether the lognormal distribution $X \sim \text{LogN}(\mu, \sigma^2)$, i.e. $\ln(X) \sim \text{N}(\mu, \sigma^2)$, is a scale distribution or not. If yes, determine whether it has any scale parameter.

**Solution**

Let $Y = cX$ for some $c > 0$. Observe that

$$\ln Y = \ln cX = \ln c + \ln X \sim \mathrm{N}(\mu + \ln c, \sigma^2).$$

For the last equation, note that if we let $Z = \ln X \sim \mathrm{N}(\mu, \sigma^2)$

$$E\left[e^{t(Z+\ln c)}\right] = e^{t\ln c} e^{\mu t + \frac{\sigma^2 t^2}{2}} = e^{t(\mu + \ln c) + \frac{\sigma^2 t^2}{2}}$$

we see that the above is the mgf of $\mathrm{N}(\mu + \ln c, \sigma^2)$. Thus we have that $Y$ has the same distribution as $X$ and so it is a scale distribution. However, we also see that it has no scale parameters.

### 6.1.2 *Raising to a Power*

**Proposition 7 (Raising to a Power)**

*Let $X$ be a crv with pdf $f_X$ and cdf $F_X$ with $F_X(0) = 0$. Let $Y = X^{\frac{1}{\tau}}$. If $\tau > 0$, then*

$$F_Y(y) = F_X(y^\tau), \quad f_Y(y) = \tau y^{\tau-1} f_X(y^\tau), \quad y > 0,$$

*while if $\tau < 0$, then*

$$F_Y(y) = 1 - F_X(y^\tau), \quad f_Y(y) = -\tau y^{\tau-1} f_X(y^\tau), \quad y > 0.$$

**Proof**

When $\tau > 0$,

$$F_Y(y) = P(Y \leq y) = P\left(X^{\frac{1}{\tau}}\right) = P\left(X \leq y^\tau\right) = F_X\left(y^\tau\right)$$

and

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} f_X\left(y^\tau\right) = \tau y^{\tau-1} f_X(y^\tau).$$

When $\tau < 0$,

$$F_Y(y) = P(Y \leq y) = P\left(X^{\frac{1}{\tau}} \leq y\right) = P\left(X \geq y^\tau\right) = \overline{F}_X(y^\tau)$$

and

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}(1 - F_X\left(y^\tau\right)) = -\tau y^{\tau-1} f_X\left(y^\tau\right).$$

□

**Example 6.1.4**

Let $X \sim \text{Exp}(\theta)$ and $Y = X^{\frac{1}{\tau}}$ for $\tau > 0$, we have

$$F_Y(y) = F_X(t^\tau) = 1 - e^{\frac{-y^\tau}{\theta}} = 1 - e^{-\left(\frac{y}{\alpha}\right)^\tau},$$

where $\alpha = \theta^{\frac{1}{\tau}}$. In particular, we have that $Y \sim \text{Wei}(\alpha, \tau)$.

### 6.1.3 *Exponentiation*

**Proposition 8 (Exponentiation Method)**

*Let $X$ be a crv with pdf $f_X$ and cdf $F_X$. Let $Y = e^X$. Then*

$$F_Y(y) = F_X(\ln y), \quad f_Y(y) = \frac{1}{y} f_X(\ln y).$$

**Proof**

We have

$$F_Y(y) = P\left(e^X \leq y\right) = P(X \leq \ln y) = F_X(\ln y)$$

and

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(\ln y) = \frac{1}{y} f_X(\ln y).$$

□

**Exercise 6.1.1 (Lognormal Distribution)**

*Let $X \sim \text{N}(\mu, \sigma^2)$. The cdf and pdf of $Y = e^X$ is*

$$F_Y(y) = F_X(\ln y) = \Phi\left(\frac{\ln y - \mu}{\sigma}\right)$$

$$f_Y(y) = \frac{1}{y} f_X(\ln y) = \frac{1}{y} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \cdot \left(\frac{\ln y - \mu}{\sigma}\right)^2}$$

### 6.1.4 Mixing Distributions

The rationale behind mixing distributions is to define an rv $X$ conditional on a second rv, say $\Theta$ (aka **mixing rv**). The mixing rv $\Theta$ can either be discrete or be continuous, which leads to two types of mixtures:

- **discrete mixture**: when $\Theta$ is discrete; and
- **continuous mixture**: when $\Theta$ is continuous.

---

**Definition 27 (Discrete Mixed Distribution)**

*Let $\Theta$ be a drv taking values on $\{\theta_1, \theta_2, \ldots, \theta_n\}$ with*

$$P(\Theta = \theta_i) = p_i > 0, \quad i = 1, \ldots, n,$$

*and the rv $Y_i := X \mid \Theta = \theta_i$ has cdf*

$$F_{Y_i}(x) = P(X \leq x \mid \Theta = \theta_i), x \in \mathbb{R}.$$

*Then $X$ is called a* ***discrete mixed distribution*** *with cdf*

$$F_X(x) = \sum_{i=1}^{n} P(X \leq x \mid \Theta = \theta_i) P(\Theta = \theta_i) = \sum_{i=1}^{n} p_i F_{Y_i}(x).$$

---

Following the above definition, by the Law of the Unconscious Statistician, we have

$$E[g(X)] = \sum_{i=1}^{n} E[g(X) \mid \Theta = \theta_i] P(\Theta = \theta_i) = \sum_{i=1}^{n} p_i E[g(Y_i)],$$

for any function $g$ such that the expectation exists. In particular, we have

$$E[X] = \sum_{i=1}^{n} p_i E[Y_i] \text{ and } E\left[X^2\right] = \sum_{i=1}^{n} p_i E\left[Y_i^2\right].$$

**Example 6.1.5**

Let $Y_i \sim \text{Exp}(i)$ for $i = 1, 2, 3$. Define $X$ to be an equal mixture of these three exponential rvs. Fidn the cdf, pdf, and mean of $X$.

**Solution**

The cdf of $X$ is

$$F_X(x) = \sum_{i=1}^{3} \frac{1}{3} F_{Y_i}(x) = \frac{(1-e^{-x}) + (1-e^{-x/2}) + (1-e^{-x/3})}{3}$$
$$= 1 - \frac{1}{3}\left(e^{-x} + e^{-\frac{x}{2}} + e^{-\frac{x}{3}}\right), x > 0.$$

The pdf of $X$ is

$$f_X(x) = \frac{1}{3}\left(e^{-x} + \frac{1}{2}e^{-\frac{x}{2}} + \frac{1}{3}e^{-\frac{x}{3}}\right), x > 0.$$

The mean of $X$ is therefore

$$E[X] = \sum_{i=1}^{3} E[Y_i] = \frac{1}{3}(1+2+3) = 2.$$

# 7 Lecture 7 Sep 27th

## 7.1 Severity Distributions — Creating Severity Distributions (Continued 2)

### 7.1.1 Mixing Distributions (Continued)

**Definition 28 (Continuous Mixture)**

*Let $\Theta$ be a crv with density $f_\Theta$, and the cdf and pdf of $X \mid \Theta = \theta$ are given by*

$$F_{X|\Theta}(x \mid \theta) = P(X \leq x \mid \Theta = \theta) \text{ and } f_{X|\Theta}(x \mid \theta) = P(X = x \mid \Theta = \theta).$$

*The unconditional distribution of $X$ is said to be a* ***continuous mixed distribution*** *with cdf and pdf*

$$F_X(x) = \int_{-\infty}^{\infty} F_{X|\Theta}(x \mid \theta) f_\Theta(\theta)\, d\theta$$
$$f_X(x) = \int_{-\infty}^{\infty} f_{X|\Theta}(x \mid \theta) f_\Theta(\theta)\, d\theta.$$

*Furthermore, for any function $H$,*

$$E[H(X)] = \int_{-\infty}^{\infty} E[H(X) \mid \Theta = \theta] f_\Theta(\theta)\, d\theta.$$

**Example 7.1.1**

Suppose that $X \mid \Lambda = \lambda$ is exponentially distributed with mean $\frac{1}{\lambda}$, and let $\Lambda$ be a gamma distributed rv with mean $\alpha/\theta$ and variance $\alpha/\theta^2$, i.e.

$$f_\Lambda(\lambda) = \frac{\theta^\alpha \lambda^{\alpha-1} e^{-\theta\lambda}}{\Gamma(\alpha)}, \lambda > 0,$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$ is the gamma function. Determine the conditional pdf of $X$.

**Solution**

We have

$$\begin{aligned} f_X(x) &= \int_0^\infty f_{X|\Lambda}(x \mid \lambda) f_\Lambda(\lambda)\, d\lambda \\ &= \int_0^\infty \lambda e^{-x\lambda} \frac{\theta^\alpha \lambda^{\alpha-1} e^{-\theta\lambda}}{\Gamma(\alpha)}\, d\lambda \\ &= \frac{\theta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^\alpha e^{-\lambda(x+\theta)}\, d\lambda \\ &= \frac{\theta^\alpha}{\Gamma(\alpha)(x+\theta)} \int_0^\infty \left(\frac{y}{x+\theta}\right)^\alpha e^{-y}\, dy \quad \text{where } y = \lambda(x+\theta) \\ &= \frac{\theta^\alpha}{\Gamma(\alpha)(x+\theta)^{\alpha+1}} \int_0^\infty y^\alpha e^{-y}\, dy \\ &= \frac{\theta^\alpha \Gamma(\alpha+1)}{\Gamma(\alpha)(x+\theta)^{\alpha+1}} = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}. \end{aligned}$$

**Proposition 9 (Total Expectation and Total Variance)**

*For any rvs X and $\Theta$, provided that the repsective expectation and variance exist, we have*

$$\begin{aligned} E[X] &= E[E[X \mid \Theta]] \\ \mathrm{Var}(X) &= E[\mathrm{Var}(X \mid \Theta)] + \mathrm{Var}(E[X \mid \Theta]) \end{aligned}$$

**Proof**

$$\begin{aligned} E[X] &= E\left(\int_X x f_{X|\Theta}(x \mid \Theta)\, dx\right) \\ &= \int_\Theta \int_X x f_{X|\Theta}(x \mid \theta) f_\Theta(\theta)\, dx\, d\theta \\ &= \int_X x \int_\Theta f_{X,\Theta}(x,\theta)\, d\theta\, dx \quad \because \text{ Fubini's Theorem} \\ &= \int_X x f_X(x)\, dx = E[X]. \end{aligned}$$

Note that

$$\mathrm{Var}(X \mid \Theta) = E[X^2 \mid \Theta] + E[X \mid \Theta]^2.$$

And so

$$\begin{aligned}
&E[\mathrm{Var}(X \mid \Theta)] + \mathrm{Var}(E[X \mid \Theta]) \\
&= E[E[X^2 \mid \Theta]] - E\left[E[X \mid \Theta]^2\right] + E\left[E[X \mid \Theta]^2\right] - E[E[X \mid \Theta]]^2 \\
&= E\left[X^2\right] - E[X]^2 = \mathrm{Var}(X)
\end{aligned}$$

□

**Example 7.1.2**

Suppose that $X \mid \Theta = \theta \sim \mathrm{Exp}(\theta)$ and $p_\Theta(\theta) = \frac{1}{3}$ for $\theta = 1, 2, 3$. Find the mean and variance of $X$.

**Solution**

The mean of $X$ is

$$E[X] = EE[X \mid \Theta] = E[\Theta] = \frac{1}{3}(1 + 2 + 3) = 2.$$

The variance of $X$ is

$$\begin{aligned}
\mathrm{Var}(X) &= E[\mathrm{Var}(X \mid \Theta)] + \mathrm{Var}(E[X \mid \Theta]) \\
&= E[\Theta^2] + \mathrm{Var}(\Theta) = 2E[\Theta^2] - E[\Theta]^2 \\
&= \frac{2}{3}(1 + 4 + 9) - 4 = \frac{28}{3} - \frac{12}{3} = \frac{16}{3}
\end{aligned}$$

**Example 7.1.3**

Suppose that $X \mid \Lambda = \lambda \sim \mathrm{Exp}(\lambda)$ and $\Lambda \sim \mathrm{Gam}(\alpha, \theta)$ with mean $\alpha\theta$ and variance $\alpha\theta^2$. Find the mean and variance of $X$.

**Solution**

The mean of $X$ is

$$E[X] = EE[X \mid \Lambda] = E[\Lambda] = \alpha\theta.$$

The variance of $X$ is

$$\begin{aligned}
\mathrm{Var}(X) &= E[\mathrm{Var}(X \mid \Lambda)] + \mathrm{Var}(E[X \mid \Lambda]) \\
&= E[\Lambda^2] + \mathrm{Var}(\Lambda) = 2\,\mathrm{Var}(\Lambda) + E[\Lambda]^2 \\
&= 2\alpha\theta^2 + \alpha^2\theta^2.
\end{aligned}$$

## 7.2 Severity Distributions — Tail of Distributions

**Definition 29 (Tail)**

*The **tail** of a distribution (usually the right tail) is the portion of the distribution corresponding to large values of the random variable.*

It is important that we understand large possible loss values as they have the greatest impact on the total losses that we may have to endure. In general, a loss rv is said to be **heavy-tailed** if it has a large probability to take large values.

Two measurements of tail weight:

- **relative**: comparing "sizes" of the tails of two distributions;
- **absolute**: classifying distributions as heavy or light-tailed.

The following is a set of criteria to measure or compare the heaviness of the tails of loss distributions:

- Existence of moments
- Limiting ratios
- Hazard rate function
- Mean excess loss function

### 7.2.1 Existence of Moments

Recall that the $k$th moment of a loss $X$ is

$$E\left[X^k\right] = \int_0^\infty x^k f_X(x)\,dx.$$

Now if $f_X$ takes on large values for large $x$, we may have $E\left[X^k\right]$ blow up to infinity, and so it is desirable to find/use some distribution with a **decaying** probability function, one at which its rate of decay is faster than the growth of $x^{-(k+1)}$.

# 8 Lecture 8 Oct 02nd

## 8.1 Severity Distributions — Tail of Distributions (Continued)

### 8.1.1 Existence of Moments (Continued)

**Example 8.1.1**

For a Pareto distribution, as $x \to \infty$, we have that $f_X(x) \sim x^{-(\alpha+1)}$, so its moments are finite if and only if $k < \alpha$.

We say that the Pareto distribution has a **power tail**.

**Example 8.1.2**

Given the transformed Gamma distribution, with pdf

$$f_X(x) = \frac{\left(\frac{x}{\theta}\right)^\alpha e^{-\frac{x}{\theta}}}{x\Gamma(\alpha)}.$$

Now as $x \to \infty$, we have

$$f_X(x) \sim x^{\alpha-1} e^{-\frac{x}{\theta}}$$

We see that the exponential term decays faster than the rate of growth of $x^{\alpha-1}$ for any $\alpha > 0$. Thus all moments of the Gamma distribution exists.

We say that the Gamma distribution has a **exponential tail**.

**Exercise 8.1.1**

*The Normal distribution has an exponential tail.*

**Definition 30 (Heavy-Tails Light-Tails)**

*We say that a distribution is a* ***heavy-tailed distribution*** *if* ***its moments only exist up to some*** $k \in \mathbb{N} \setminus \{0\}$.

*We say that a distribution is a* ***light-tail distribution*** *if* ***its moments exist for all*** $k \in \mathbb{N} \setminus \{0\}$.

The actual definition, or should I say notion, of tail-heaviness comes from talking about the boundedness of the tail of the distribution, with reference to the exponential distribution. If a distribution has a tail that has greater value than the tail of the exponential distribution, then we say that the distribution has a heavy-tail.

**Note**

*We may also use the mgf to determine if a distribution has a heavy or light tail; the inexistence of the kth moment implies the inexistence of the mgf, i.e. if the mgf does not exist, then the moments of the distribution is only finite up to some* $k \in \mathbb{N} \setminus \{0\}$.

### 8.1.1.1 Limiting Ratio: Survival Functions

**Definition 31 (Limiting Ratio)**

*The* ***limiting ratio*** *of* ***two survival functions*** *is used to compare the heaviness of tails of the two losses. Consider two losses X and Y, and consider the limit of the ratio*

$$\lim_{x \to \infty} \frac{\overline{F}_X(x)}{\overline{F}_Y(x)}.$$

*If the limit does not exist, we say that the comparison is inconclusive. Otherwise, we have 3 cases:*

- *If* $c = 0$, *then* $\overline{F}_X(x)$ *decays faster than* $\overline{F}_Y(x)$ *as* $x \to \infty$, *i.e. Y has a heavier tail than X;*
- *If* $0 < c < \infty$, *then* $\overline{F}_X(x)$ *and* $\overline{F}_Y(x)$ *decays at the smae rate, as* $x \to \infty$, *i.e. X and Y have similar tails;*
- *If* $c = \infty$, *then* $\overline{F}_X(x)$ *decays slower than* $\overline{F}_Y(x)$ *as* $x \to \infty$, *i.e. X has a heavier tail than Y;*

*where we let*

$$c := \lim_{x \to \infty} \frac{\overline{F}_X(x)}{\overline{F}_Y(x)}$$

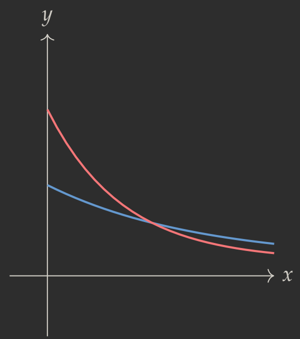


Figure 8.1: Limiting Ratio

**Note**

*Not all distributions have an explicit survival function, but they will always have a pdf/pmf. Fortunately, by* ***L'Hôpital's Rule****, the above definition can be applied to the pdfs of $X$ and $Y$, i.e.*

$$c = \lim_{x\to\infty} \frac{\bar{F}_X(x)}{\bar{F}_Y(x)} = \lim_{x\to\infty} \frac{-f_X(x)}{-f_Y(x)} = \lim_{x\to\infty} \frac{f_X(x)}{f_Y(x)}$$

---

**Example 8.1.3**

Show that the Pareto distribution has a heavier tail than the Gamma distribution using limiting ratio.

**Solution**

Let $X \sim \text{Pareto}(\alpha, \theta)$ and $Y \sim \text{Gam}(\tau, \lambda)$. We have

$$c = \lim_{x\to\infty} \frac{f_X(x)}{f_Y(x)} = \lim_{x\to\infty} \frac{\frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}}{\frac{x^{\tau-1}e^{-\frac{x}{\lambda}}}{\lambda^\tau\Gamma(\tau)}} = \alpha\theta^\alpha\lambda^\tau\Gamma(\tau) \lim_{x\to\infty} \frac{e^{\frac{x}{\lambda}}}{x^{\tau-1}(x+\theta)^{\alpha+1}}$$

Since the exponential term grows faster than the term in the denominator, we have $c = \infty$, i.e. $X$ has a heavier tail than $Y$, as required.

**Example 8.1.4**

For two losses $X$ and $Y$, suppose that $f_X(x) = \frac{2}{\pi(1+x^2)}$ and $f_Y(x) = \frac{1}{(1+x^2)}$ for $x > 0$. Compare the tail heaviness of the two losses.

**Solution**

Notice that

$$c = \lim_{x\to\infty} \frac{f_X(x)}{f_Y(y)} = \lim_{x\to\infty} = \frac{2}{\pi} < \infty,$$

i.e. $X$ and $Y$ have similar tails.

### 8.1.1.2 *Hazard Rate*

RECALL Definition 12. We had

$$h(x) = \frac{f(x)}{\bar{F}(x)} = -\frac{d}{dx} \ln \bar{F}(x),$$

$$h_X(x)\Delta x \approx P(X \leq x + \Delta x \mid X > x)$$

and the hazard rate function relates to the survival function as

$$\bar{F}(x) = e^{-\int_{-\infty}^{x} h(y)\,dy}.$$

Notice that

- if the hazard rate function is a **decreasing** function, that implies that the probability of the occurrence of $X \leq x + \Delta x$ decreases given $X > x$, as $x$ increases, i.e. it is more likely that we have $X > x + \Delta x \mid X > x$. So $X$ has a **heavy tail**.
- if the hazard rate function is a **increasing** function, that implies that the probability of the occurrence of $X \leq x + \Delta x$ increases given $X > x$, as $x$ increases, i.e. it is less likely that $X > x + \Delta x \mid X > x$. So $X$ has a **light tail**.

**Definition 32 (Decreasing and Increasing Failure Rates)**

*Let $X$ be a loss with hazard rate function $h_X$. We say that*[1]

- *$X$ or $F_X$ has a* ***decreasing failure rate (DFR)*** *if $h_X$ is decreasing;*
- *$X$ or $F_X$ has a* ***increasing failure rate (IFR)*** *if $h_X$ is increasing.*

[1] The following source claims that the **failure rate** and hazard rate are, in fact, not always interchangable terms: `https://nomtbf.com/2013/11/difference-hazard-failure-rate/`. Perhaps this is worth looking into.

**Note**

*Consequently,*

- *Distributions that have a DFR are heavy-tailed;*
- *Distributions that have an IFR are light-tailed.*

**Proposition 10 (Exponential has Constant Hazard Rate)**

*The exponential distribution has a constant hazard rate.*

**Proof**

The pdf and survival function of $X \sim \mathrm{Exp}(\lambda)$ is

$$f_X(x) = \lambda e^{-\lambda x} \text{ and } \bar{F}_X(x) = e^{-\lambda x},$$

respectively. Thus the hazard rate of $X$ is

$$h(x) = \frac{f_X(x)}{\bar{F}_X(x)} = \lambda,$$

which is a fixed value. □

**Note**

*We say that the exponential distribution is the only distribution which is said to have both DFR and IFR.*[2]

[2] **Why?**

**Example 8.1.5**

Let $X \sim \text{Pareto}(\alpha, \theta)$ with $f_X(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}$ and $\bar{F}_X(x) = \frac{\theta^\alpha}{(x+\theta)^\alpha}$. Determine whether $X$ has a DFR or IFR.

**Solution**

The hazard rate function of $X$ is

$$h_X(x) = \frac{f_X(x)}{\bar{F}_X(x)} = \frac{\frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}}{\frac{\theta^\alpha}{(x+\theta)^\alpha}} = \frac{\alpha}{x+\theta}.$$

It is clear that $h_X$ is a decreasing function, and so $X \sim \text{Pareto}(\alpha, \theta)$ has a DFR, i.e. it is heavy-tailed.

It is not always easy to get the survival function. The following is an alternative approach to finding out if the hazard rate function is increasing or decreasing.

**Proposition 11 (Ratio Comparison for DFR/IFR)**

*Let $X$ be an rv, and*[3]

$$s(x) = \frac{f_X(x+y)}{f_X(x)}.$$

1. *If $s(x)$ is increasing in $x$ for every $y$, then $X$ has a DFR;*
2. *If $s(x)$ is decreasing in $x$ for every $y$, then $X$ has an IFR.*

[3] **Any bounds on $y$?**

**Proof**

We shall prove for one case as the other will follow analogously. Notice that

$$h_X(x) = \frac{f_X(x)}{\overline{F}_X(x)} = \frac{f_X(x)}{\int_x^\infty f_X(y)\,dy} = \frac{1}{\int_0^\infty \frac{f_X(x+y)}{f_X(x)}\,dy}$$

by a change of variable in the last equality. We notice that if $\frac{f_X(x+y)}{f_X(x)}$ is increasing, then $h_X(x)$ will be decreasing, and so $X$ has a DFR. ☐

---

**Example 8.1.6**

Let $X \sim \mathrm{Gam}(\alpha, \theta)$ with $\alpha > 1$. Determine whether $X$ is a DFR or IFR distribution.

**Solution**

The cdf of $X$ is

$$f_X(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^\alpha \Gamma(\alpha)}.$$

The survival function of $X$ is not explicit, and so we should use Proposition 11. We have

$$\frac{f_X(x+y)}{f_X(x)} == \frac{\frac{(x+y)^{\alpha-1}e^{-\frac{x+y}{\theta}}}{\theta^\alpha\Gamma(\alpha)}}{\frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^\alpha\Gamma(\alpha)}} = \left(\frac{x+y}{x}\right)^{\alpha-1} e^{-\frac{y}{\theta}} = \left(1+\frac{y}{x}\right)^{\alpha-1} e^{-\frac{y}{\theta}}.$$

To try to determine if it is increasing or decreasing, we calculate the second derivative of the ratio:

$$\frac{d}{dx}\left(1+\frac{y}{x}\right)^{\alpha-1} e^{-\frac{y}{\theta}} = y(\alpha-1)\left(1+\frac{y}{x}\right)^{\alpha-2} e^{-\frac{y}{\theta}}.$$

It is important to note that $y$ is not completely free: it is bounded below by $-x$, as if $y < -x$, then $x+y < 0$, and $f$ is undefined at these values. Also, if $y = -x$, then the ratio is simply a constant, and we cannot use Proposition 11 to reach a conclusion. To be able to use Proposition 11, we must have $y > -x$. In this case, it is clear that the ratio is increasing as $x$ increases. Thus $X$ has an IFR.

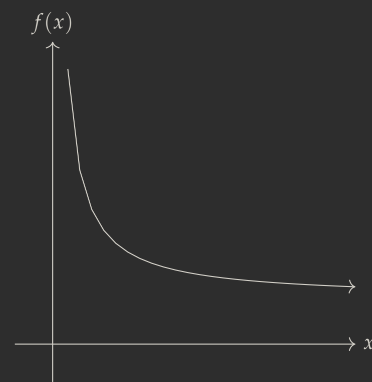


Figure 8.2: Graph of $(1+\frac{y}{x})^{\alpha-1} e^{-\frac{y}{\theta}}$ for $y > -x$ and $x > 0$.

# 9 Lecture 9 Oct 11th

## 9.1 Severity Distributions — Tail of Distributions (Continued 2)

### 9.1.1 Mean Excess Loss

**Definition 33 (Excess Loss Random Variable)**

*For a loss rv $X$, we define the* ***excess loss rv*** *as*

$$T_d = X - d \mid X > d, \quad d > 0.$$

*The survival function of $T_d$ is*

$$\begin{aligned}\bar{F}_{T_d}(x) &= P(T_d > x) = P(X - d > x \mid X > d) \\ &= \frac{P(X > x + d)}{P(X > d)} = \frac{\bar{F}_X(x+d)}{\bar{F}_X(d)}.\end{aligned}$$

As defined before in Definition 22,

**Definition (Mean Excess Loss)**

*The* ***mean excess loss*** *(or* ***mean residual life****) function is defined as*

$$e_X(d) = E[T_d] = \int_0^\infty \bar{F}_{T_d}(x)\,dx = \frac{\int_0^\infty \bar{F}_X(x+d)\,dx}{\bar{F}_X(d)} = \frac{\int_d^\infty \bar{F}_X(y)\,dy}{\bar{F}_X(d)}$$

Essentially, the mean excess loss is the average payment in excess of the threshold $d$, given that the loss exceeds the threshold.

**Definition 34 (Increasing and Decreasing Mean Residual Lifetime)**

*Given a loss rv $X$,*

1. *we say $X$ or $F_X$ is an* ***increasing mean residual lifetime (IMRL)*** *if $e_X(x)$ is increasing in $x$;*
2. *we say $X$ or $F_X$ is an* ***decreasing mean residual lifetime (DMRL)*** *if $e_X(x)$ is decreasing in $x$.*

**Note**

- *IMRL distributions are* ***heavy-tailed****;*
- *DMRL distributions are* ***light-tailed****.*

*The reason of this claim should be rather clear from the context of $e_X(x)$: if $e_X(x)$ is increasing with $x$, then we expect that the survival probability of $T_d$ to be greater, and so the tail should be a heavy one. The following proposition clarifies this notion.*

**Proposition 12 (Relation between DFR/IFR and IMRL/DMRL)**

*A DFR rv is IMRL, and an IFR rv is a DMRL.*

**Proof**

Suppose $X$ has a DFR. The mean excess loss of $X$ is

$$e_X(d) = \frac{\int_0^\infty \bar{F}_X(x+d)\,dx}{\bar{F}_X(d)} = \int_0^\infty \frac{\bar{F}_X(x+d)}{\bar{F}_X(d)}\,dx.$$

Note that by the relationship between the survival function and the hazard rate[1],

$$\frac{\bar{F}_X(x+d)}{\bar{F}_X(d)} = \frac{e^{-\int_0^{x+d} h_X(y)\,dy}}{e^{-\int_0^d h_X(y)\,dy}} = e^{-\int_d^{x+d} h_X(y)\,dy} = e^{-\int_0^x h_X(z+d)\,dz}.$$

Since $X$ has a DFR, $h_X$ is decreasing, and thus $\frac{\bar{F}_X(x+d)}{\bar{F}_X(d)}$ is increasing. Thus $e_X(d)$ is increasing and so $X$ is a IMRL, as required. THe argument is similar for $IFL$ being a $DMRL$. □

[1] We use the hazard rate here because it is provided by the assumption.

**Example 9.1.1**

Let $X \sim \text{Wei}(\theta, \tau)$. Determine whether $X$ is DMRL or IMRL.

**Solution**

Since

$$f_X(x) = \frac{\tau x^{\tau-1} e^{-\left(\frac{x}{\theta}\right)^\tau}}{\theta^\tau}$$

and from an earlier example, we have

$$\bar{F}_X(x) = e^{-\left(\frac{x}{\theta}\right)^\tau}$$

Then the hazard rate is

$$h_X(x) = \frac{f_X(x)}{\bar{F}_X(x)} = \frac{\tau}{\theta^\tau} x^{\tau-1}.$$

Now if $\tau \geq 1$, then $h_X(x)$ is an increasing function, and so $X$ has an IFR, i.e. $X$ is a DMRL. if $0 < \tau \leq 1$, then $h_X(x)$ is a decreasing function, and so $X$ has a DFR, i.e. $X$ is an IMRL.

**Example 9.1.2**

Consider a loss $X$ with $f_X(x) = (1 + 2x^2)e^{-2x}$ for $x > 0$.

1. Determine $h_X(x)$.

2. Determine $e_X(x)$.

3. Find $\lim_{x \to \infty} h_X(x)$ and $\lim_{x \to \infty} e_X(x)$.

4. Show that $X$ is DMRL but not IFR.

**Solution**

Since both $h_X(x)$ and $e_X(x)$ require the survival function, we shall first derive that. Observe that[2]

$$\begin{aligned}
\bar{F}_X(x) &= \int_x^\infty (1 + 2y^2)e^{-2y}\,dy = \frac{1}{2}e^{-2x} + 2\left[\int_x^\infty y^2 e^{-2y}\,dy\right] \\
&= \frac{1}{2}e^{-2x} + 2\left[-\frac{1}{2}y^2 e^{-2y}\Big|_x^\infty + \int_x^\infty y e^{-2y}\,dy\right] \\
&= \frac{1}{2}e^{-2x} + x^2 e^{-2x} + 2\left[-\frac{1}{2}y e^{-2y}\Big|_x^\infty + \frac{1}{2}\int_x^\infty e^{-2y}\,dy\right] \\
&= \frac{1}{2}e^{-2x} + x^2 e^{-2x} + x e^{-2x} + \frac{1}{2}e^{-2x} \\
&= (x^2 + x + 1)e^{-2x}.
\end{aligned}$$

[2] It is highly recommended that one gets really used to using integration by parts, to the point that you do not have to repeatedly write down what the $u$ and $dv$ are explicitly every time.

1. It is clear that

$$h_X(x) = \frac{1+2x^2}{1+x+x^2}$$

2. By its definition, we have that

$$e_X(x) = \frac{\int_x^\infty \bar{F}_X(y)\,dy}{\bar{F}_X(x)},$$

and so we need to solve for the integral in the numerator. Using pieces from our derivation of $\bar{F}_X(x)$, we obtain

$$\begin{aligned}
&\int_x^\infty (1+y+y^2)e^{-2y}\,dy \\
&= \frac{1}{2}e^{-2x} + \frac{1}{2}xe^{-2x} + \frac{1}{4}e^{-2x} + \frac{1}{2}x^2e^{-2x} + \frac{1}{2}xe^{-2x} + \frac{1}{4}e^{-2x} \\
&= \left(1+x+\frac{1}{2}x^2\right)e^{-2x}.
\end{aligned}$$

Thus

$$e_X(x) = \frac{1+x+\frac{1}{2}x^2}{1+x+x^2}.$$

3. The answers are straightforward[3]

$$\begin{aligned}
\lim_{x\to\infty} h_X(x) &= \lim_{x\to\infty} \frac{\frac{1}{x^2}+2}{1+\frac{1}{x}+\frac{1}{x^2}} = 2 \\
\lim_{x\to\infty} e_X(x) &= \lim_{x\to\infty} \frac{\frac{1}{x^2}+\frac{1}{x}+\frac{1}{2}}{\frac{1}{x^2}+\frac{1}{x}+1} = \frac{1}{2}
\end{aligned}$$

[3] **Find out why did we calculate these values.**

4. First, observe that

$$\begin{aligned}
e_X'(x) &= \frac{(1+x)\left(1+x+x^2\right) - (1+2x)\left(1+x+\frac{1}{2}x^2\right)}{(1+x+x^2)} \\
&= -\frac{x+\frac{1}{2}x^2}{(1+x+x^2)^2},
\end{aligned}$$

and we see that $e_X'(x) < 0$ for $x > 0$. Thus $X$ has a DMRL. For $h_X(x)$,

$$\begin{aligned}
h_X'(x) &= \frac{4x\left(1+x+x^2\right) - (1+2x)\left(1+2x^2\right)}{\left(1+x+x^2\right)^2} \\
&= \frac{2x^2+2x-1}{x^4+2x^3+3x^3+2x+1}.
\end{aligned}$$

It may appear as if $h_X'(x)$ is positive, seeing that $x^4$ should dominate. However, notice that the discriminant is positive:[4]

$$2^2 - 4(2)(-1) = 12 > 0,$$

and so the numerator has a root, i.e. there are critical points on $h_X(x)$. In fact, equating the said numerator to 0, we can obtain that $x = -\frac{1}{2} + \sqrt{\frac{3}{4}}$ (the other case is ruled out as $x > 0$). Since $h'_X(x)$ looks as if it is increasing, let's try out some values of $x$ for $0 < x < \sqrt{\frac{3}{4}} - \frac{1}{2}$. In particular, notice that

$$h_X\left(\frac{1}{10}\right) = \frac{102}{111} \approx 0.9198$$
$$h_X\left(\frac{1}{5}\right) = \frac{27}{31} \approx 0.8710$$

but $\frac{1}{10} < \frac{1}{5}$, and so we notice that $X$ is not IFR.

[4] Lecture notes simply threw the values 1 and $\frac{1}{2}$ for $x$ almost out of nowhere. While the result seems harmless, firstly, $x \neq 0$, since $x > 0$. In fact, since the critical point is $\sqrt{\frac{3}{4}} - \frac{1}{2} \approx 0.366$, $\frac{1}{2}$ is a value that comes after the critical point, so we would not have been able to verify without trying and failing numerous times, especially since the critical point is an irrational value.

Here, we are smart and equipped with the knowledge that by solving the first derivative for $x$ by equating to 0 allows us to find these critical points, which is indicative of a change from positive to negative, or vice versa, slope for $h_X(x)$.

# 10 Lecture 10 Oct 16th

## 10.1 Severity Distributions — Policy Adjustments (Continued)

Insurance policies contain various **adjustments** to soften the amount that insurers have to pay, to minimize moral hazards, and for various other reasons. In this section, we shall introduce some common policy adjustments.

In the following definitions, suppose that $X$ is our ground-up loss rv, and $H$ a function incurred by the adjustment.

**Definition 35 (Policy Limit)**

*A fixed level $u > 0$ is called a **policy limit** if, provided that there are no other adjustments, the insurer shall pay*[1]

$$H(X) = \min\{X, u\} := X \wedge u = \begin{cases} X & X \leq u \\ u & X \geq u \end{cases}.$$

[1] Now that this definition uses the symbol $\wedge$ for denoting a policy limit, I shall refrain from using the same symbol in proofs, unless if the context is clear.

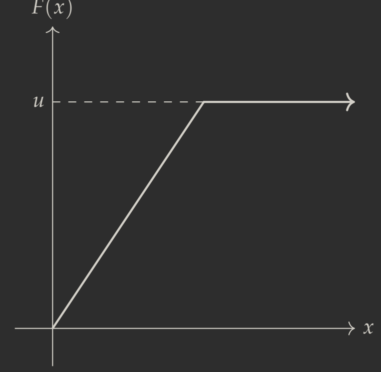

Figure 10.1: Typical graph of a policy limit, without other adjustments.

**Note**

- *A policy limit protects the insurer from overly large losses.*
- *This is **noteworthy**: in practice, a policy limit may refer to **the maximum amount paid** by the insurer, but in this course, it is the **maximum loss** coverred by the insurer.*

**Definition 36 (Ordinary Deductible)**

*A fixed level $d > 0$ is called an* ***ordinary deductible*** *if, given that there are no other adjustments, the insurer pays*

$$Y = H(X) = (X-d)_+ = X \vee d = \begin{cases} 0 & X < d \\ X - d & X \geq d \end{cases}$$

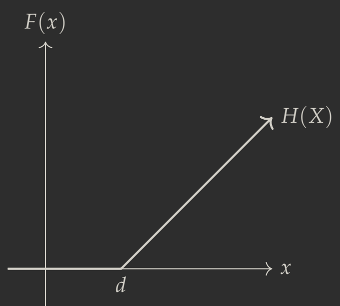


Figure 10.2: Graph of a policy with ordinary deductible without any other adjustments.

**Note**

- *For any given loss, the first $d$ dollars falls on the insured.*
- *It is a protection against frequent small claims.*

**Definition 37 (Franchise Deductible)**

*A fixed level $d > 0$ is called a* ***Franchise Deductible*** *if, given that there are no other adjustments, the insurer pays*

$$\begin{aligned} H(X) = X \cdot \mathbb{1}_{\{X>d\}} &= \begin{cases} 0 & X \leq d \\ X & X > d \end{cases} \\ &= (X-d)\mathbb{1}_{\{X>d\}} + d \cdot \mathbb{1}_{\{X>d\}} \\ &= (X-d)_+ + d \cdot \mathbb{1}_{\{X>d\}} \end{aligned}$$

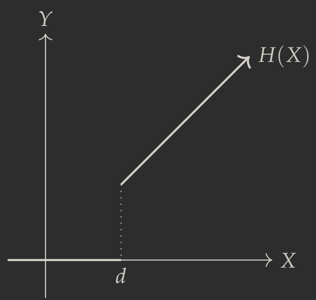


Figure 10.3: Graph of a policy with Franchise deductible without any other adjustments.

**Note**

- *This differs from the ordinary deductible in that twhen the loss exceeds $d$, the deductible is waived and the* ***full loss is paid*** *by the insurer.*
- *We are not concerned with whether the payment goes out or not at $X = d$ in this course.*[2]

[2] In the event that a problem of such a nature comes out in either exercises or exams, the point will be explicitly stated.

**Remark**

*This is not a good adjustment as it is prone to* ***moral hazard****.*

**Definition 38 (Coinsurance)**

*A fixed rate $\alpha \in [0,1]$ is called a* ***coinsurance factor*** *if, given that there are no other adjustments, the insurer pays*

$$H(X) = \alpha X.$$

*For any given loss, the* ***insurer pays a proportion*** $100\alpha\%$ *of the loss amount the remaining $100(1-\alpha)\%$ falls on the insured.*

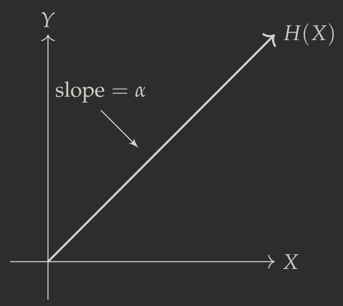


Figure 10.4: Graph of a policy with coinsurance without any other adjustments.

### 10.1.1 Application Order for Multiple Adjustments

If an insurance policy has more than one adjustment, we assume the adjustments in the following order:

- Policy limit (if any)
- Policy/ordinary deductible (if any)
- Coninsurance (if any)

**Note**

- *These transformations are not necessarily commutative, so the order must be obeyed.*
- *This ordering is optimal, i.e. it covers for all possible combinations, i.e. any other ways of adjustment can be expressed in this form.*[3]
- *If $d$ is a deductible and $u$ the policy limit, we must have that $d < u$, since if $u < d$, then the insurer will only pay the maximum amount $u$ if the loss exeeds $d$, which is absurd. Therefore, for all of the cases that we shall consider, we will always assume, and safely so, that $d < u$.*

[3] Claimed by lecturer. Require example.

Applying the ordering, we have

$$X \to X \wedge u \to [(X \wedge u) - d]_+ \to \alpha[(X \wedge u) - d]_+$$

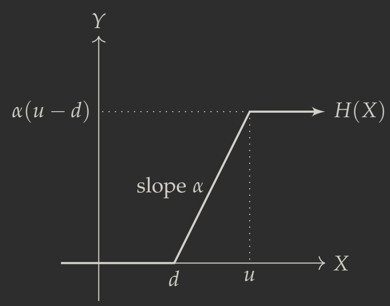


Figure 10.5: Graph of $H(X) = \alpha[(X \wedge u) - d]_+$.

and so

$$H(X) = \alpha[(X \wedge u) - d]_+ = \begin{cases} 0 & X < d \\ \alpha(X-d) & d \leq X < u \\ \alpha(u-d) & X \geq u \end{cases}$$

For the case of applying **Franchise deductible** instead of ordinary deductible, we have

$$X \to X \wedge u \to (X \wedge u)\mathbb{1}_{\{X>d\}} \to \alpha(X \wedge u) \cdot \mathbb{1}_{\{X>d\}}$$

Notice that $X \wedge u \to (X \wedge u)\mathbb{1}_{\{X>d\}}$, since $X \wedge u > d$ is simply $X > d$ as $u > d$ by assumption. We have that for the case where we consider the Franchise deductible instead of an ordinary deductible,

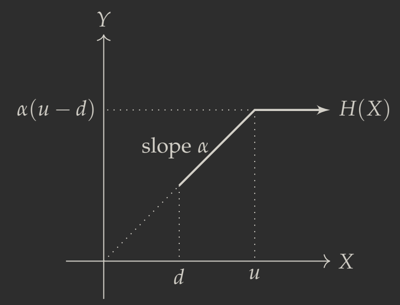


Figure 10.6: Graph of $H(X) = \alpha(X \wedge u) \cdot \mathbb{1}_{\{X>d\}}$.

$$H(X) = \alpha(X \wedge u) \cdot \mathbb{1}_{\{X>d\}} = \begin{cases} 0 & X < d \\ \alpha X & d \leq X < u \\ \alpha u & X \geq u \end{cases}$$

### 10.1.2 ★ Reporting Methods

When we consider the amount paid by the insurer, we typically consider (and distinguish) between two types of reporting methods.

- **Loss basis**: $Y_L$ = amount paid per loss
- **Payment basis**: $Y_P$ = amount paid per payment

It is sensible that

$$Y_P = Y_L \mid Y_L > 0. \tag{10.1}$$

The above relationship shows that we can retrieve $Y_P$ from $Y_L$, i.e. $Y_L$ holds more information than $Y_P$. This makes sense; a loss may occur, but the insurer may not have to pay for the loss. For example, if the incurred loss is below a given deductible level.

#### 10.1.2.1 Loss Basis

- Each loss is recorded, i.e. each loss has an entry, even if the amount paid is 0.
- For a policy limit $u$, ordinary deductible $d$ with $d < u$ [4], and

[4] It would be silly if $d > u$.

coinsurance factor $\alpha$, we have

$$Y_L = \alpha[(X \wedge u) - d]_+.$$

- For a policy with limit $u$, Franchise deductible $d$ with $d < u$, and coinsurance factor $\alpha$, we have

$$Y_L = \alpha(X \wedge u)\mathbb{1}_{\{X>d\}}.$$

- Note that in the presence of a deductible $d$, it is **usually the case** that $Y_L$ has a probability mass at 0, i.e.

$$P(Y_L = 0) = P(X \le d) = F_X(d) > 0.$$

In this case, $Y_P > 0$ almost surely.

#### 10.1.2.2 *Payment Basis*

- Only **non-zero** payments of the insurer are included, and so not every loss will have an entry. Here, we see that $Y_P$ leaves that information behind.[5]

[5] It is still useful to reporting purely on the financial effects of the claims.

- $Y_P$ does not have a probability mass at 0 (**Why**?), i.e.

$$P(Y_P = 0) = 0,$$

or equivalently,

$$Y_P > 0.$$

**Example 10.1.1**

Let $X$ be the ground-up loss rv and assume that there is an ordinary deductible of 5 applied to the loss. The following table is a typical example illustrating how $Y_L$ and $Y_P$ works.

| $X$ | 3 | 2 | 5 | 7 | 9 | 10 |
|---|---|---|---|---|---|---|
| $Y_L$ | 0 | 0 | 0 | 2 | 4 | 5 |
| $Y_P$ | NA | NA | NA | 2 | 4 | 5 |

Table 10.1: Example illustrating the relationship between $Y_P$ and $Y_L$.

# 11 Lecture 11 Oct 18th

## 11.1 Severity Distribution — Policy Adjustments (Continued 2)

### 11.1.1 Distribution & Moments of $Y_P$ and $Y_L$

It suffices for us to closely study $Y_L$ due to the following proposition:

**Proposition 13 ($Y_P$ is completely determined by $Y_L$)**

*The survival function and moments of $Y_P$ are given by*

$$\bar{F}_{Y_P}(y) = \begin{cases} 1 & y < 0 \\ \frac{\bar{F}_{Y_L}(y)}{\bar{F}_{Y_L}(0)} & y \geq 0 \end{cases}$$

*and*

$$E\left[Y_P^k\right] = \frac{E\left[Y_L^k\right]}{\bar{F}_{Y_L}(0)}, \quad k = 1, 2, \ldots.$$

**Proof**

Using the definition of a survival function, we have

$$\bar{F}_{Y_P}(x) = P(Y_P > x) = P(Y_L > x \mid Y_L > 0) = \begin{cases} 1 & x < 0 \\ \frac{\bar{F}_{Y_L}(x)}{\bar{F}_{Y_L}(0)} & x \geq 0 \end{cases}.$$

Consequently,

$$e\left[Y_P^k\right] = k\int_0^\infty x^{k-1}\bar{F}_{Y_P}(x)\,dx = k\int_0^\infty x^{k-1}\frac{\bar{F}_{Y_L}(x)}{\bar{F}_{Y_L}(0)}\,dx = \frac{E\left[Y_L^k\right]}{\bar{F}_{Y_L}(0)}.$$

□

**Note**

*Proposition 13 tells us that it suffices to discover the distribution of $Y_L$, since it completely determines $Y_P$,*

**Remark**

- *If there is a deductible $d > 0$, then the distributions of $Y_P$ and $Y_L$ are **usually**[1] different.*
- *If $Y_L$ has no mass point at 0, i.e. $\bar{F}_{Y_L}(0) = 1$, then $Y_P$ nand $Y_L$ have the same distribution.*

[1] This depends on the distribution of $X$ and $Y_L$.

### 11.1.2 Some Important Identities

The following proposition is important for us to venture forward.

**Proposition 14 (★★ Expected Value of the Policy Adjustments)**

*Consider a non-negative rv $X$ and $d > 0$. Then*

1. *We have*
$$E[X] = E[[X-d]_+] + E[X \wedge d].$$
2. *For $k = 1, 2, \ldots,$*
$$E\left[X^k\right] = \int_0^\infty x^{k-1}\bar{F}_X(x)\,dx.$$
3. *For $k = 1, 2, \ldots,$*
$$E\left[(X \wedge d)^k\right] = \int_0^d kx^{k-1}\bar{F}_X(x)\,dx.$$
4. *For $k = 1, 2, \ldots,$*
$$E\left[[X-d]_+^k\right] = \int_d^\infty k(x-d)^{k-1}\bar{F}_X(x)\,dx.$$
5. *For $k = 1, 2, \ldots,$ and $\bar{F}_X(d) > 0$,*
$$E\left[(X-d)^k \mid X > d\right] = \frac{E\left[[X-d]_+^k\right]}{\bar{F}_X(d)} = \frac{\int_d^\infty k(x-d)^{k-1}\bar{F}_X(x)\,dx}{\bar{F}_X(d)}.$$

*In particular, we have an alternate way to derive the mean excess value*

$$e_X(d) = E[X - d \mid X > d] = \frac{\int_d^\infty \overline{F}_X(x)\,dx}{\overline{F}_X(d)}.$$

**Proof**

1. For this identity, notice that

$$[X-d]_+ + (X \wedge d) = \begin{cases} 0 + X & X \leq d \\ X - d + d & X > d \end{cases} = X.$$

The result follows from linearity of $E$.

2. We have proved this earlier on, but it shall be re-proved for exercise, variety, and ease of reference.[2]

$$\begin{aligned} E\left[X^k\right] &= \int_0^\infty x^k f_X(x)\,dx = \int_0^\infty x^k \frac{d}{dx} F_X(x)\,dx \\ &= \int_0^\infty x^k dF_X(x) = \int_0^\infty x^k d(1 - \overline{F}_X(x)) \\ &= -\int_0^\infty x^k d\overline{F}_X(x) \\ &= -x^k \overline{F}_X(x)\Big|_0^\infty + k\int_0^\infty x^{k-1}\overline{F}_X(x)\,dx \quad \because \text{IBP} \\ &= k\int_0^\infty x^{k-1}\overline{F}_X(x)\,dx, \end{aligned}$$

under the assumption that $\overline{F}_X(x)$ decays faster than $x^k$ [3].

3. Using a similar argument as in the earlier part of the last proof, and by the Law of the Unconscious Statistician, we can arrive at

$$E\left[(X \wedge u)^k\right] = -\int_0^\infty (x \wedge u)^k\, d\overline{F}_X(x).$$

To proceed, use integration by parts as follows[4]:

$$\begin{aligned} u &= (x \wedge u)^k & v &= \overline{F}_X(x) \\ du &= d(x \wedge u)^k & dv &= d\overline{F}_X(x) \end{aligned}$$

We get

$$E\left[(X \wedge u)^k\right] = -(x \wedge u)^k \overline{F}_X(x)\Big|_0^\infty + \int_0^\infty \overline{F}_X(x)\,d(x \wedge u)^k,$$

[2] **Important**: There are two rules that you must use, and it does not depend on any of your existing knowledge as an undergrad whatsoever.
(a) Notice that $\frac{d}{dx} F_X(x) = f_X(x) \implies -d\overline{F}_X(x) = f_X(x)\,dx$;
(b) While using integration by parts, let $dv = d\overline{F}_X(x)$ so that $v = \overline{F}_X(x)$.
Forget any one of these are prepare to be screwed over.

[3] How unlikely is this, I do not know.

[4] This is hopeless. **If you can't remember this**, or somehow make some sense of this monster (without going through a few lectures on Lesbesgue or Riemann-Stieljes integration), **you're screwed.**

and $(x \wedge u)^k \bar{F}_X(x) \Big|_0^\infty = 0$. Next, it is a "fact" that

$$d(x \wedge u)^k = \begin{cases} kx^{k-1}\,dx & x < u \\ 0 & x \geq u \end{cases}$$

[5]Since $x > u$ gives us a 0 term, we are left with

$$E\left[(X \wedge u)^k\right] = \int_0^u kx^{k-1}\bar{F}_X(x)\,dx.$$

[5] This is yet another monstrosity that makes no sense whatsoever for one that has only taken basic Calculus classes and introductory Analysis. **Remember this "fact"** or **get screwed over**. The only "sense" that I can come up with right now is to consider the cases of when $x < u$ and when $x \geq u$, and determine what $x \wedge u$ should be in these cases, and provide some baseless rationalization.

4. Using the Law of the Unconscious Statistician and Item 2, we have

$$E\left[[X-d]_+^k\right] = k\int_0^\infty (x-d)_+^k \bar{F}_X(x)\,dx = k\int_d^\infty (x-d)^k \bar{F}_X(x)\,dx.$$

5. Notice once and for all that

$$\begin{aligned} E\left[(X-d)^k \mid X > d\right] &= \frac{E\left[(X-d)^k \mathbb{1}_{\{X>d\}}\right]}{\bar{F}_X(d)} \\ &= \frac{E\left[[X-d]_+^k\right]}{\bar{F}_X(d)} = \frac{\int_d^\infty k(x-d)^k \bar{F}_X(x)\,dx}{\bar{F}_X(d)}. \end{aligned}$$

□

---

**Example 11.1.1**

Consider a ground-up loss $X$ with pdf

$$f_X(x) = 0.0005, \quad 0 \leq x \leq 20.$$

Solve for

1. $\bar{F}_X(x)$;
2. $E[X \wedge 10]$ and $E[X \wedge 25]$;
3. $\mathrm{Var}(X \wedge 10)$;
4. $E\left[[X-10]_+\right]$;
5. $E\left[[X-10]_+^2\right]$; and
6. $e_X(10)$.

**Solution**

1. We have

$$F_X(x) = \int_0^x 0.005y\,dy = 0.0025x^2$$

and so

$$\bar{F}_X(x) = \begin{cases} 1 & x < 0 \\ 1 - 0.0025x^2 & 0 \leq x \leq 20 \\ 0 & x > 20 \end{cases}.$$

2. Using our identities,

$$E[X \wedge 10] = \int_0^{10} \bar{F}_X(x)\,dx = 10 - \frac{5}{6000}(10)^3 = \frac{55}{6},$$

and

$$E[X \wedge 25] = \int_0^{20} \bar{F}_X(x)\,dx = 25 - \frac{5}{6000}(25)^3 = \frac{40}{3}.$$

3. To get $\mathrm{Var}(X \wedge 10)$, we first need the 2nd moment of $X \wedge 10$:

$$\begin{aligned} E\left[(X \wedge 10)^2\right] &= 2\int_0^{10} x\bar{F}_X(x)\,dx = 2\left[\frac{1}{2}x^2 - \frac{5}{8000}x^4\right]_0^{10} \\ &= 100 - \frac{25}{2} = \frac{175}{2}. \end{aligned}$$

Thus

$$\mathrm{Var}\,(X \wedge 10) = \frac{175}{2} - \left(\frac{55}{6}\right)^2 = \frac{125}{36}.$$

4. We have that

$$\begin{aligned} E\left[[X - 10]_+\right] &= \int_{10}^{20} 1 - \frac{1}{400}x^2\,dx \\ &= 10 - \frac{1}{1200}(8000 - 1000) = \frac{25}{6}. \end{aligned}$$

5. We have that

$$\begin{aligned} E\left[[X - 10]_+^2\right] &= 2\int_{10}^{20} (x - 10)\left(1 - \frac{1}{400}x^2\right)dx \\ &= 2\int_{10}^{20} \left(-10 + x + \frac{1}{40}x^2 - \frac{1}{400}x^3\right)dx \\ &= 2\left[-100 + \frac{300}{2} + \frac{7000}{120} - \frac{150000}{1600}\right] \\ &= \frac{175}{6} \end{aligned}$$

6. We have

$$e_X(10) = \frac{\int_{10}^{20} \left(1 - \frac{1}{400}x^2\right) dx}{1 - \frac{1}{400}(10)^2} = \frac{10 - \frac{7000}{1200}}{\frac{3}{4}} = \frac{50}{9}$$

#### 11.1.2.1 *Application of Proposition 14*

*Policy Limit* If a policy limit $u$ is the only adjustment in a contract, then

$$Y_L = X \wedge u \quad \text{and} \quad Y_P = X \wedge u \mid X > 0.$$

Since in most cases $X > 0$, we have that $\bar{F}_X(0) = 1$, and so $Y_L = Y_P = X \wedge u$.

- The survival function is

$$\bar{F}_{Y_P}(y) = \bar{F}_{Y_L}(y) = P(X \wedge u > y) = \begin{cases} 1 & y < 0 \\ \bar{F}_X(y) & 0 \leq y < u \\ 0 & y \geq u \end{cases}$$

- The expected value is

$$E[Y_P] = E[Y_L] = E[X \wedge u] = \int_0^u \bar{F}_X(x)\, dx.$$

- The second moment is

$$E\left[Y_P^2\right] = E\left[Y_L^2\right] = E\left[(X \wedge u)^2\right] = 2\int_0^u x\bar{F}_X(x)\, dx$$

*Ordinary Deductible* If an ordinary deductible $d$ is the only adjustmnet in a contract, then

$$Y_L = [X - d]_+ = (X - d)\mathbb{1}_{\{X>d\}}$$

and

$$Y_P = [X - d]_+ \mid [X - d]_+ > 0 = (X - d)\mathbb{1}_{\{X>d\}} \mid X > d = X - d \mid X > d$$

In most cases, since $\bar{F}_X(d) = P(X > d) < 1$ as it is $P(X < d) \neq 0$, the distribution of $Y_L$ and $Y_P$ differs.

- The survival function is

$$\bar{F}_{Y_L}(y) = P([X-d]_+ > y) = \begin{cases} 1 & y < 0 \\ \bar{F}_X(y+d) & y \geq 0 \end{cases}$$

and

$$\bar{F}_{Y_P}(y) = \frac{\bar{F}_{Y_L}(y)}{\bar{F}_{Y_L}(0)} = \begin{cases} 1 & y < 0 \\ \frac{\bar{F}_X(d+y)}{\bar{F}_X(d)} & y \geq 0 \end{cases}$$

for $Y_L$ and $Y_P$ respectively.

- The mean is

$$E[Y_L] = E[[X-d]_+] = \int_d^\infty \bar{F}_X(x)\,dx$$

and

$$E[Y_P] = E[X-d \mid X > d] = \frac{E[(X-d)\mathbb{1}_{\{X>d\}}]}{\bar{F}_X(d)} = \frac{\int_d^\infty \bar{F}_X(x)\,dx}{\bar{F}_X(d)}$$

for $Y_L$ and $Y_P$ respectively.

- The second moment is

$$E\left[Y_L^2\right] = E\left[[X-d]_+^2\right] = 2\int_d^\infty x\bar{F}_X(x)\,dx$$

and

$$E\left[Y_P^2\right] = \frac{E\left[Y_L^2\right]}{\bar{F}_{Y_L}(0)} = \frac{2\int_d^\infty x\bar{F}_X(x)\,dx}{\bar{F}_X(d)}$$

for $Y_L$ and $Y_P$ respectively.

# 12 Lecture 12 Oct 23rd

## 12.1 Severity Distribution — Policy Adjustments (Continued 3)

### 12.1.1 Some Important Identities (Continued)

#### 12.1.1.1 Application of Proposition 14 (Continued)

*Policy Limit + Ordinary Deductible* If there is a policy limit $u$ and an ordinary deductible $d$ with $u > d$, then

$$Y_L = [(X \wedge u) - d]_+ = \begin{cases} 0 & X < d \\ X - d & d \leq X < u \\ u - d & X \geq u \end{cases}$$

and so its survival function is[1]

$$\begin{aligned} \bar{F}_{Y_L}(y) &= P(Y_L > y) = P([(X \wedge u) - d]_+ > y) \\ &= P(\max\{0, (X \wedge u) - d\} > y) \\ &= \begin{cases} 1 & y < 0 \\ P((X \wedge u) - d > y) & y \geq 0 \end{cases} \\ &= \begin{cases} 1 & y < 0 \\ P(X > y + d) & 0 \leq y < u - d \\ P(u > y + d) & y \geq u - d \end{cases} \\ &= \begin{cases} 1 & y < 0 \\ \bar{F}_X(y + d) & 0 \leq y < u - d \\ 0 & y \geq u - d \end{cases} \end{aligned}$$

[1] **Important**: Pay attention to the notion here: we are using the actual definition of a deductible to arrive at an explicit solution.

Consequently, its moments are

$$\begin{aligned} E\left[Y_L^k\right] &= \int_0^\infty ky^{k-1}\bar{F}_{Y_L}(y)\,dy \\ &= \int_0^{u-d} ky^{k-1}\bar{F}_X(y+d)\,dy \\ &= \int_d^u k(y-d)^{k-1}\bar{F}_X(y)\,dy \end{aligned}$$

For $Y_P$, we have

$$\begin{aligned} Y_P &= Y_L \mid Y_L > 0 = [(X \wedge u) - d]_+ \mid [(X \wedge u) - d]_+ > 0 \\ &= (X \wedge u) - d \mid X > d. \end{aligned}$$

Thus by Proposition 13,

$$\bar{F}_{Y_P}(y) = \begin{cases} 1 & y < 0 \\ \frac{\bar{F}_{Y_L}(y)}{\bar{F}_{Y_L}(0)} & 0 \le y < u-d \\ 0 & y \ge u-d \end{cases}$$

Using the same proposition, the moments of $Y_P$ are

$$E\left[Y_P^k\right] = \frac{E\left[Y_L^k\right]}{\bar{F}_X(d)} = \frac{\int_d^u k(y-d)^{k-1}\bar{F}_X(y)\,dy}{\bar{F}_X(d)}.$$

*Franchise Deductible* Given a Franchise Deductible $d > 0$, we have

$$Y_L = X \cdot \mathbb{1}_{\{X>d\}}.$$

Its survival function is

$$\begin{aligned} \bar{F}_{Y_L}(y) = P(X \cdot \mathbb{1}_{\{X>d\}} > y) &= \begin{cases} 1 & y < 0 \\ P(X \cdot \mathbb{1}_{\{X>d\}} > y) & y \ge 0 \end{cases} \\ &= \begin{cases} 1 & y < 0 \\ \bar{F}_X(d) & 0 \le y < d \\ \bar{F}_X(y) & y \ge d \end{cases}. \end{aligned}$$

For the case when $0 \leq y < d$, it is clear that in order for $X \cdot \mathbb{1}_{\{X>d\}} > y$, we first need $X > d$. Now the moments of $Y_L$ are

$$\begin{aligned} E\left[Y_L^k\right] &= \int_0^\infty k y^{k-1} \bar{F}_{Y_L}(y)\, dy \\ &= \bar{F}_X(d) \int_0^d k y^{k-1}\, dy + \int_d^\infty k y^{k-1} \bar{F}_X(y)\, dy \\ &= d^k \bar{F}_X(d) + \int_d^\infty k y^{k-1} \bar{F}_X(y)\, dy. \end{aligned}$$

Now observe that

$$Y_P = Y_L \mid Y_L > 0 = X \cdot \mathbb{1}_{\{X>d\}} \mid X \cdot \mathbb{1}_{\{X>d\}} > 0 = X \mid X > d$$

Again, using Proposition 13,

$$\bar{F}_{Y_P}(y) = \frac{\bar{F}_{Y_L}(y)}{\bar{F}_{Y_L}(0)} = \begin{cases} 1 & y < d \\ \frac{\bar{F}_X(y)}{\bar{F}_X(d)} & y \geq d \end{cases}$$

and its moments are

$$\begin{aligned} E\left[Y_P^k\right] &= \frac{E\left[Y_L^k\right]}{\bar{F}_X(d)} = \frac{d^k \bar{F}_X(d) + \int_d^\infty k y^{k-1} \bar{F}_X(y)\, dy}{\bar{F}_X(d)} \\ &= d^k + \frac{\int_d^\infty k y^{k-1} \bar{F}_X(y)\, dy}{\bar{F}_X(d)}. \end{aligned}$$

*Coinsurance* Let $\hat{Y}_L$ be the amount to-be-paid per loss without coinsurance. Let $\alpha \in (0,1)$ be a coinsurance factor. Applying coinsurance for adjustment, we have that

$$Y_L = \alpha \hat{Y}_L.$$

The survival function of $Y_L$ is

$$\bar{F}_{Y_L}(y) = P\left(\alpha \hat{Y}_L > y\right) = P\left(\hat{Y}_L > \frac{y}{\alpha}\right) = \begin{cases} 1 & y < 0 \\ \bar{F}_{\hat{Y}_L}\left(\frac{y}{\alpha}\right) & y \geq 0 \end{cases}$$

and its moments are

$$E\left[Y_L^k\right] = E\left[\alpha^k \hat{Y}_L{}^k\right] = \alpha^k E\left[\hat{Y}_L{}^k\right].$$

Then for $Y_P$, we have that its survival function is

$$\bar{F}_{Y_P}(y) = \frac{\bar{F}_{Y_L}(y)}{\bar{F}_{Y_L}(0)} = \frac{\bar{F}_{\hat{Y}_L}\left(\frac{y}{\alpha}\right)}{\bar{F}_{\hat{Y}_L}(0)}, \quad y \geq 0,$$

and its moments

$$E\left[Y_P^k\right] = \frac{E\left[Y_L^k\right]}{\bar{F}_{\hat{Y}_L}(0)} = \frac{\alpha^k E\left[\hat{Y_L}^k\right]}{\bar{F}_{\hat{Y}_L}(0)}.$$

**Example 12.1.1**

The cdf of a ground-up loss $X$ is given by

$$F_X(x) = 1 - \left(1 - \frac{x}{800}\right)^2, \quad 0 \leq x \leq 800.$$

Assuming a policy limit of 600, an ordinary deductible of 200, and a coinsurance facotr of 0.8, determine the cdf of $Y_L$ and the expected amount paid per loss $E\left[Y_L\right]$.

**Solution**

We are given

$$Y_L = 0.8[(X \wedge 600) - 200]_+.$$

The survival function of $Y_L$ is

$$\begin{aligned}
\bar{F}_{Y_L}(y) &= P\left(\max\{0, (X \wedge 600) - 200\} > \frac{y}{0.8}\right) \\
&= \begin{cases} 1 & y < 0 \\ P\left(X \wedge 600 > \frac{5y}{4} + 200\right) & y \geq 0 \end{cases} \\
&= \begin{cases} 1 & y < 0 \\ P\left(X > \frac{5y}{4} + 200\right) & 0 \leq y < 0.8(600 - 200) = 320 \\ 0 & y \geq 320 \end{cases} \\
&= \begin{cases} 1 & y < 0 \\ \bar{F}_X\left(\frac{5y}{4} + 200\right) & 0 \leq y < 320 \\ 0 & y \geq 320 \end{cases}
\end{aligned}$$

Thus the cdf of $Y_L$ is

$$\bar{F}_{Y_L}(y) = \begin{cases} 0 & y < 0 \\ 1 - \bar{F}_X\left(\frac{5y}{4} + 200\right) & 0 \leq y < 320 \\ 1 & y \geq 320 \end{cases}.$$

The expected amount paid per loss is

$$E\left[Y_L\right] = 0.8 \int_{200}^{600} \left(1 - \frac{x}{800}\right)^2 dx = \frac{260}{3}$$

**Example 12.1.2**

Consider a ground-up loss $X$ with a Franchise deductible $d$. You are given that

- 15% of the losses are below the Franchise deductible $d$,
- the mean excess loss $e_X(d) = 50$,
- the expected amount paid per loss is 51.

Determine the value of $d$.

**Solution**

We are given

$$\begin{aligned} Y_L &= X \cdot \mathbb{1}_{\{X>d\}} \\ P(X < d) &= 0.15 \end{aligned}$$

Thus $P(X > d) = 0.85$. We have

$$\bar{F}_{Y_L}(y) = \begin{cases} 1 & y < 0 \\ \bar{F}_X(d) & 0 \le y < d \\ \bar{F}_X(y) & y \ge d \end{cases} = \begin{cases} 1 & y < 0 \\ 0.85 & 0 \le y < d \\ \bar{F}_X(y) & y \ge d \end{cases}.$$

We are given

$$50 = e_X(d) = \frac{\int_d^\infty \bar{F}_X(y)\,dy}{\bar{F}_X(d)},$$

and so

$$\int_d^\infty \bar{F}_X(y)\,dy = 50 * 0.85 = 42.5.$$

We are given

$$\begin{aligned} 51 = E[Y_L] &= \int_0^\infty \bar{F}_X(y)\,dy = d\bar{F}_X(d) + \int_d^\infty \bar{F}_X(y)\,dy \\ &= 0.85d + 42.5. \end{aligned}$$

Thus

$$d = \frac{8.5}{0.85} = 10.$$

# 13 Lecture 13 Oct 25th

## 13.1 Severity Distribution — Policy Adjustments (Continued 4)

By introducing policy adjustments, it is within our interest to determine if the introduced adjustments have helped to eliminate the expected propotion of loss.

**Definition 39 (Loss Elimination Ratio)**

*The **loss elimination ratio**, denoted as* LER, *is the ratio of which loss has been mitigated, or eliminated, as a result of policy adjustments, and it is given by*

$$\mathrm{LER} = \frac{E[X - Y_L]}{E[X]} = 1 - \frac{E[Y_L]}{E[X]},$$

*where* $\frac{E[Y_L]}{E[X]}$ *corresponds to the percentage of loss retained by the insurer.*

**Example 13.1.1**

For a policy that has only an ordinary deductible, i.e. $Y_L = [X - d]_+$, we have

$$\mathrm{LER} = 1 - \frac{E([X-d]_+)}{E[X]} = 1 - \frac{E[X] - E[X \wedge d]}{E[X]} = \frac{E[X \wedge d]}{E[X]}.$$

**Example 13.1.2**

Consider a ground-up loss $X \sim \mathrm{Pareto}(\alpha, \theta)$ with $\alpha = 2$ and $\theta = 1000$.

1. Calculate the LER if an ordinary deductible of 500 is applied.
2. What is the required value of $d$ to eliminate 20% of the loss?

**Solution**

1. Note that
$$\bar{F}_X(x) = \frac{\theta^\alpha}{(x+\theta)^\alpha}.$$
Now
$$\begin{aligned} E[X] &= \int_0^\infty \bar{F}_X(x)\,dx = \theta^\alpha \int_0^\infty \frac{1}{(x+\theta)^\alpha}\,dx \\ &= \frac{\theta^\alpha}{1-\alpha} \cdot \frac{1}{(x+\theta)^{\alpha-1}}\bigg|_0^\infty = \frac{\theta}{\alpha-1}. \end{aligned}$$
and
$$\begin{aligned} E[X \wedge d] &= \int_0^d \bar{F}_X(x)\,dx = \frac{\theta^\alpha}{1-\alpha} \cdot \frac{1}{(x+\theta)^{\alpha-1}}\bigg|_0^d \\ &= \frac{\theta^\alpha}{1-\alpha}\left(\frac{1}{(d+\theta)^{\alpha-1}} - \frac{1}{\theta^{\alpha-1}}\right) \\ &= \frac{\theta}{\alpha-1}\left(1 - \left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right). \end{aligned}$$
Thus
$$\text{LER} = \frac{E[X \wedge d]}{E[X]} = \left(1 - \left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right) = \frac{1}{3}.$$
In other words, $\frac{1}{3}$ is mitigated by setting an ordinary deductible of 500.

2. In this case, let $\text{LER} = 0.2 = \frac{1}{5}$. Then
$$\frac{1}{5} = 1 - \frac{1000}{d+1000} \iff \frac{1000}{d+1000} = \frac{4}{5} \iff d = 250$$

## 13.2 Frequency Distributions — Basic Frequency Distributions

Recall from our Collective Risk Model that
$$S = \sum_{i=1}^N X_i,$$
where

$X_i \equiv$ size of the $i^{\text{th}}$ claim, modelled by severity distributions

$N \equiv$ a nonnegative integer-valued rv that represents the number of claims, modelled by frequency distributions

**Definition 40 (Counting Distributions and RVs)**

*A nonnegative rv, usually represented by $N$, is called a* ***counting rv*** *and its distribution is called a* ***counting distirbution****.*

**Note**

*For this section, the pgf is important.*

*Importance of PGF* Given $G(t) = E\left[t^N\right] = \sum_{k=0}^{\infty} t^K p_k$, provided that the moments exist, we have

$$\begin{aligned} G^{(n)}(t) = \frac{d^n}{dt^n} G(t) &\overset{(*)}{=} E\left[\prod_{i=1}^{n} (N - i + 1) t^{N-n}\right] \\ &= \sum_{k=0}^{\infty} \prod_{i=1}^{n} (k - i + 1) t^{k-n} p_k \\ &\overset{(**)}{=} \sum_{k=n}^{\infty} \prod_{i=1}^{n} (k - i + 1) t^{k-n} p_k \end{aligned}$$

Also for $(*)$: The derivation is

$$\begin{aligned} G_N^{(n)}(t) &= \frac{d^n}{dt^n} G_N(t) \\ &= \sum_{k=0}^{\infty} \frac{d^n}{dt^n} t^k p_k \\ &= \sum_{k=0}^{\infty} k(k-1)\dots(k-n+1) t^{k-n} p_k \\ &= E\left[\prod_{k=1}^{n} (N - k + 1) t^{N-n}\right] \end{aligned}$$

where $(*)$ is because the moments exist, and $(**)$ is because for $k = 0, 1, \dots, n-1$, the product $\prod_{i=1}^{n} (k - i + 1) = 0$.

We can obtain the pmf of $N$ from the pgf by

$$\begin{aligned} G^{(n)}(0) &= \sum_{k=n}^{\infty} \prod_{i=1}^{n} (k - i + 1) t^{k-n} p_k \Big|_{t=0} \\ &= \prod_{i=1}^{n} (n - i + 1) p_n = n! p_n \end{aligned} \tag{13.1}$$

where we notice in Equation (13.1) that only the $n^{\text{th}}$ term survives as $t^{n-n} = 1$.

**Factorial Moments**[1] can be obtained by

$$G^{(n)}(1) = E\left[\prod_{i=1}^{n} (N - i + 1)\right], \quad n = 1, 2, 3, \dots.$$

In particular, we have that

$$G'(1) = E[N] \text{ and } G''(1) = E[N(N-1)] = E(N^2) - E(N)$$

and so

$$\operatorname{Var}(N) = G''(1) + G'(1) - G'(1)^2.$$

[1] **Definition 41 (Factorial Moments from PGF)**

*We can obtain the* ***factorial moments*** *of an rv $X$ from its pgf. In particular,*

$$G^{(n)}(1) = E\left[\prod_{i=1}^{n} (X - i + 1)\right]$$

*where $n \in \mathbb{N} \setminus \{0\}$.*

### 13.2.1 Frequency Distributions

#### 13.2.1.1 Poisson Distribution

**Definition 42 (Poisson Distribution)**

*A counting rv $N$ is said to have a* ***Poisson distribution*** *with parameter $\lambda$, and denote $N \sim \mathrm{Poi}(\lambda)$, if it has the pmf*

$$p_k = P(N = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots .$$

**Remark**

*We can easily verify that the Poisson distribution is indeed a probability distribution, by noticing that*

$$\sum_{k=0}^{\infty} p_k = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1$$

*where we used the* ***Taylor expansion*** $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

**Proposition 15 (PGF, Mean, and Variance of Poisson Distribution)**

*For $N \sim \mathrm{Poi}(\lambda)$, its pgf is*

$$G(t) = e^{\lambda(t-1)},$$

*and its mean and variance are*

$$E(N) = \mathrm{Var}(N) = \lambda.$$

**Proof**

Notice that

$$G(t) = E\left[t^N\right] = \sum_{k=0}^{\infty} t^k p_k = \sum_{k=0}^{\infty} \frac{e^{-\lambda}(t\lambda)^k}{k!} = e^{-\lambda} e^{t\lambda} = e^{\lambda(t-1)}.$$

Thus

$$E[N] = G'(1) = \lambda \text{ and } G''(1) = \lambda^2,$$

and so

$$\text{Var}(N) = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

□

### Proposition 16 (Sum of Independent Poisson RVs)

*If $N_1, N_2, \ldots, N_m$ are* ***independent*** *Poisson rvs with parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$ respectively, then*

$$N = \sum_{i=1}^{m} N_i \sim \text{Poi}\left(\sum_{i=1}^{m} \lambda_i\right)$$

### Proof

Using the pgf method for $N$, we see that

$$\begin{aligned} G(t) &= e\left[t^N\right] = E\left[t^{\sum_{i=1}^m N_i}\right] = \prod_{i=1}^{m} E\left[t^{N_i}\right] \\ &= \prod_{i=1}^{m} e^{\lambda_i(t-1)} = e^{(t-1)\sum_{i=1}^m \lambda_i} \end{aligned}$$

which is the pgf of Poi $\left(\sum_{i=1}^m \lambda_i\right)$ as required. □

# 14 Lecture 14 Oct 30th

## 14.1 Frequency Distribution — Basic Frequency Distributions (Continued)

### 14.1.1 Frequency Distributions (Continued)

#### 14.1.1.1 Poisson Distribution (Continued)

**Proposition 17 (Splitting a Poisson Distribution)**

*Suppose that the total number of claim arrivals follows $N \sim \mathrm{Poi}(\lambda)$. There are $m$ distinct types of claims. Given a claim occurs, it is of type $i$ with probability $p_i$ such that*

$$p_1 + \ldots + p_m = 1.$$

*Then, for each fixed $i = 1, \ldots, m$, the number of claims of type $i$, $N_i \sim \mathrm{Poi}(\lambda p_i)$. Furthermore, $N_1, N_2, \ldots, N_m$ are independent.*

**Note**

*The above proposition can be visualized using a tree.*

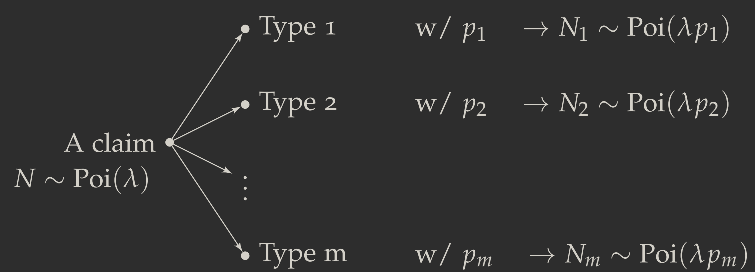


Figure 14.1: Visualization of Proposition 17

**Proof**

We shall use **mathematical induction** on $m$, for the statement "$N_1, N_2, \ldots, N_m$ are independent". $N_i \sim \mathrm{Poi}(\lambda p_i)$ will follow from the induction step.

For $m = 1$, there is nothing to prove. It suffices to prove for $m = 2$, since we may think of the problem as

$$\text{Type 1} \quad \underbrace{\text{Type 2} \quad \text{Type 3} \ldots \text{Type m}}_{\text{Type 2}'}.$$

**Case $m = 2$** Suppose $N = N_1 + N_2 \sim \mathrm{Poi}(\lambda)$. To show that $N_1$ and $N_2$ are independent, a relation which we denote as $N_1 \perp N_2$, we need to show

$$P(N_1 = k_1, N_2 = k_2) = P(N_1 = k_1)P(N_2 = k_2), \tag{14.1}$$

which is a defining property of independence.

Firstly, note that if given sets $A \subset B$, we have

$$P(A) = P(A \cap B).$$

With that,

$$\begin{aligned}
P(N_1 = k_1, N_2 = k_2) &= P(\overbrace{N_1 = k_1, N_2 = k_2}^{A}, \overbrace{N_1 + N_2 = k_1 + k_2}^{B}) \\
&= P(A \mid B)P(B) \\
&= \binom{k_1 + k_2}{k_1} p_1^{k_1} p_2^{k_2} \cdot \frac{e^{-\lambda} \lambda^{k_1 + k_2}}{(k_1 + k_2)!} \\
&= \frac{(k_1 + k_2)!}{k_1! k_2!} p_1^{k_1} p_2^{k_2} \cdot \frac{e^{-\lambda(1)} \lambda^{k_1 + k_2}}{(k_1 + k_2)!} \\
&= e^{-\lambda(p_1 + p_2)} \cdot \frac{(\lambda p_1)^{k_1}}{k_1!} \cdot \frac{(\lambda p_2)^{k_2}}{k_2!} \\
&= \frac{e^{-\lambda p_1} (\lambda p_1)^{k_1}}{k_1!} \cdot \frac{e^{-\lambda p_2} (\lambda p_2)^{k_2}}{k_2!}
\end{aligned} \tag{14.2}$$

Thus, the marginal distribution of $N_1$

$$\begin{aligned} P(N_1 = k_1) &= \sum_{k_2=0}^{\infty} P(N_1 = k_1, N_2 = k_2) \\ &= \frac{e^{-\lambda p_1}(\lambda p_1)^{k_1}}{k_1!} e^{-\lambda p_2} \sum_{k_2=0}^{\infty} \frac{(\lambda p_2)^{k_2}}{k_2!} \\ &= \frac{e^{-\lambda p_1}(\lambda p_1)^{k_1}}{k_1!} e^{-\lambda p_2} e^{\lambda p_2} \\ &= \frac{e^{-\lambda p_1}(\lambda p_1)^{k_1}}{k_1!} \end{aligned}$$

which is the pmf of $\text{Poi}(\lambda p_1)$. The marginal distribution of $N_2$ is similar. It is clear from Equation (14.2) that we have Equation (14.1). The result then follows from induction. □

**Example 14.1.1 (Thinning Property of the Poisson Distribution)**

The number of claims of a portfolio follows $\text{Poi}(\lambda)$. The severity of ground-up loss follows $\text{Unif}(0, b)$. The insurer would like to impose an ordinary deductible $d$ and a policy limit $u$ such that

$$0 < d < u < b.$$

What is the frequency distribution of **positive payments**?

**Solution**

Let Type 1 be the case where $X < d$ and Type 2 be $X > d$. Since the severity of the ground-up loss follows $\text{Unif}(0, b)$, the probability of an occurrence of Type 2 is

$$1 - \frac{d}{b} = \frac{b-d}{b}.$$

By Proposition 17, we have that the frequency distribution of positive payments, i.e. Type 2, follows $\text{Poi}\left(\lambda \frac{b-d}{b}\right)$.

### 14.1.1.2 *Binomial Distribution*

**Definition 43 (Binomial Distribution)**

*A counting rv $N$ is said to have a* ***binomial distribution*** *with parameters $q \in (0, 1)$ and $m \in \mathbb{N} \setminus \{0\}$, written as $N \sim \text{Bin}(q, m)$, if it has the* *pmf*

$$p_k = P(N = k) = \binom{m}{k} q^k (1-q)^{m-k}, \ k = 0, 1, 2, \dots.$$

**Remark**

*It is easy to verify that this is a valid probability distribution, since*

$$\sum_{k=0}^{m} \binom{m}{k} q^k (1-q)^{m-k} = (q + 1 - q)^m = 1^m = 1$$

*by the* ***binomial theorem****.*

**Note**

- *When $m = 1$, the distribution is called a* ***Bernoulli*** *rv with mean $q$.*
- *The binomial distribution is a* ***bounded*** *rv, since it has a fixed number of trials.*

**Proposition 18 (PGF of Binomial Distribution)**

*Let $N \sim \mathrm{Bin}(q, m)$. Its pgf is given by*

$$G(t) = (1 - q + tq)^m.$$

*Moreover, its mean and variance are*

$$E[N] = mq \text{ and } \mathrm{Var}(N) = mq(1-q)$$

*respectively.*

**Proof**

We have

$$G(t) = \sum_{k=0}^{m} \binom{m}{k} (tq)^k (1-q)^{m-k} = (1 - q + tq)^m$$

The mean is, therefore,

$$G'(1) = mq(1 - q + (1)q)^{m-1} = mq,$$

and its variance

$$\begin{aligned}\mathrm{Var}(N) &= G''(1) + G'(1) - G'(1)^2 \\ &= m(m-1)q^2 + mq - m^2q^2 = mq(1-q).\end{aligned}$$

□

**Proposition 19 (Sum of Independent Binomial RVs)**

*If $N_1, \ldots, N_n$ are independent and $N_i \sim \mathrm{Bin}(q, m_i)$ for $i = 1, \ldots, n$, then*

$$N = \sum_{i=1}^n N_i \sim \mathrm{Bin}\left(q, \sum_{i=1}^n m_i\right).$$

**Proof**

We shall use the pgf to prove this, instead of using the mgf (which is the common approach).

$$\begin{aligned}G_N(t) = E\left[t^N\right] = E\left[t^{\sum_{i=1}^n N_i}\right] &\overset{(*)}{=} \prod_{i=1}^n E\left[t^{N_i}\right] \\ &= \prod_{i=1}^n (1-q+tq)^{m_i} = (1-q+tq)^{\sum\limits_{i=1}^n m_i}\end{aligned}$$

Thus $N = \sum_{i=1}^n N_i \sim \mathrm{Bin}\left(q, \sum\limits_{i=1}^n m_i\right)$. □

**Note**

*As a result of Proposition 19, if we have a sequence of Bernoulli trials, each with the same "success" probability $q$, call each of them $I_i$, then*

$$N = \sum_{i=1}^m I_i \sim \mathrm{Bin}(q, m)$$

*Consequently, it becomes rather silly how easy it is we can get the mean* *and variance of N:*

$$E[N] = E\left[\sum_{i=1}^{m} I_i\right] = \sum_{i=1}^{m} E[I_i] = mq$$
$$\text{Var}(N) = \text{Var}\left(\sum_{i=1}^{m} I_i\right) = \sum_{i=1}^{m} \text{Var}(I_i) = mq(1-q)$$

# 15 Lecture 15 Nov 01st

## 15.1 Frequency Distribution — Basic Frequency Distributions (Continued 2)

### 15.1.1 Frequency Distributions (Continued 2)

#### 15.1.1.1 Negative Binomial Distribution

**Definition 44 (Negative Binomial Distribution)**

*A counting rv $N$ is said to have a* ***negatiev binomial distribution*** *with parameters $\beta > 0$ and $r > 0$, denoted $N \sim \mathrm{NB}(\beta, r)$, if it has the pmf*

$$p_k = P(N = k) = \binom{k+r-1}{k} \left( \frac{1}{1+\beta} \right)^r \left( \frac{\beta}{1+\beta} \right)^k, k = 0, 1, 2, \ldots$$

**Remark**

*Note that*

$$\binom{k+r-1}{k} = \frac{\Gamma(k+r)}{k!\Gamma(r)} = \frac{(k+r-1)!}{k!(r-1)!},$$

*where the later equality follows if $r \in \mathbb{N} \setminus \{0\}$.*

**Note**

- *When $r = 1$, we can also write the pmf of $\mathrm{NB}(\beta, 1)$ as the pmf of the* ***geometric distribution****:*

$$p_k = \frac{1}{1+\beta} \left( \frac{\beta}{1+\beta} \right)^k, k = 0, 1, 2, \ldots.$$

- *To verify that the negative binomial distribution is a valid probability* *distribution, we need the following identity:*

$$(1-x)^{-r} = \sum_{k=0}^{\infty} \binom{k+r-1}{k} x^k,$$

*which is proven as follows:*

**Exercise 15.1.1**
*Verify that the negative binomial distribution is a valid probability distribution.*

**Proof**

We shall use the **Taylor expansion** of $(1-x)^{-r}$.

$$\begin{aligned}(1-x)^{-r} &= 1 + (-1)(-r)(1-x)^{-r-1}\Big|_{x=0} x \\ &\quad + \frac{r}{2}(-1)(-r-2)(1-x)^{-r-2}\Big|_{x=0} x^2 + \ldots \\ &= 1 + rx + \frac{r(r+1)}{2}x^2 + \frac{r(r+1)(r+2)}{3!}x^3 + \ldots \\ &= \sum_{k=0}^{\infty} \frac{r(r+1)\ldots(r+k-1)}{k!} x^K \\ &= \sum_{k=0}^{\infty} \binom{k+r-1}{k} x^k.\end{aligned}$$

□

- *The negative binomial distribution is an **unbounded** rv, and can take all natural numbers sans* 0.

*Interpretation* Consider an experiment with independent trails, of which each has only two possible outcomes: success with probability $\frac{1}{1+\beta}$, and failure with probability $1 - \frac{1}{1+\beta} = \frac{\beta}{1+\beta}$. Let $N$ denote the number of failures until reaching the $r^{\text{th}}$ success.

**Proposition 20 (PGF of the Negative Binomial Distribution)**

*Let* $N \sim \mathrm{NB}(\beta, r)$. *Its pgf is thus*

$$G(t) = [1 - \beta(t-1)]^{-r}.$$

*Moreover, its mean and variance are*

$$E[N] = r\beta \text{ and } \mathrm{Var}(N) = r\beta(1+\beta),$$

*respectively.*

**Note**

*Note that the proof for getting the pgf is similar to how we can verify that $N$ is a probability (same case as in earlier counting distributions).*

**Proof**

Using the Taylor Expansion $(1-x)^{-r} = \sum_{k=0}^{\infty} \binom{k+r-1}{k} x^k$, we have

$$\begin{aligned} G(t) &= \sum_{k=0}^{\infty} t^k p_k = \left(\frac{1}{1+\beta}\right)^r \sum_{k=0}^{\infty} \binom{k+r-1}{k} \left(\frac{t\beta}{1+\beta}\right)^k \\ &= \left(\frac{1}{1+\beta}\right)^r \left(1 - \frac{t\beta}{1+\beta}\right)^{-r} = [1-\beta(t-1)]^{-r}. \end{aligned}$$

Consequently, the mean is

$$E[N] = G'(1) = -r(-\beta) = r\beta$$

and variance is

$$\begin{aligned} \mathrm{Var}(N) &= G''(1) + G'(1) - G'(1)^2 \\ &= -r(-r-1)\beta^2 + r\beta - r^2\beta^2 = r\beta(1+\beta) \end{aligned}$$

□

**Proposition 21 (Negative Binomial from Poisson Conditioned on Gamma)**

*Let* $N \mid \Lambda = \lambda \sim \mathrm{Poi}(\lambda)$ *and* $\Lambda \sim \mathrm{Gam}(\alpha, \theta)$. *Then*

$$N \sim \mathrm{NB}(\theta, \alpha).$$

**Note**

*We may also write* $N \mid \Lambda = \lambda \sim \mathrm{Poi}(\lambda)$ *as* $N \mid \Lambda \sim \mathrm{Poi}(\Lambda)$.

**Proof**

We shall prove this statement by finding the pgf of $N$, which identifies the distribution. Note that

$$G_N(t) = E\left[t^N\right] \overset{\text{Proposition 9}}{=} E\left[E\left[t^N \mid \Lambda\right]\right] \overset{(*)}{=} E\left[e^{\Lambda(t-1)}\right],$$

where $(*)$ **requires further clarification**. Now since $\Lambda \sim \mathrm{Gam}(\alpha, \theta)$, and $M_\Lambda(t) = E\left[e^{t\Lambda}\right] = (1-\theta t)^{-\alpha}$, it follows that

$$G_N(t) = [1 - \theta(t-1)]^{-\alpha}.$$

Thus $N \sim \mathrm{NB}(\theta, \alpha)$. □

**Proposition 22 (Combining Negative Binomial Distributions)**

*If $\{N_i\}_{i=1}^n$ is a sequence of independent rvs, and $N_i \sim \mathrm{NB}(\beta, r_i)$. Then*

$$N = \sum_{i=1}^n N_i \sim \mathrm{NB}\left(\beta, \sum_{i=1}^n r_i\right).$$

**Proof**

We shall, again, use the pgf. We have

$$\begin{aligned} G_N(t) = E\left[t^N\right] &\overset{(*)}{=} \prod_{i=1}^n E\left[t^{N_i}\right] = \prod_{i=1}^n G_{N_i}(t) \\ &= \prod_{i=1}^n [1 - \beta(t-1)]^{r_i} = [1 - \beta(t-1)]^{-\sum_{i=1}^n r_i}, \end{aligned}$$

where $(*)$ is by independence of the rvs, and the last equality is thanks to $\beta$ being fixed for all the rvs. This completes the proof. □

### 15.1.2 $(a, b, n)$ Classes

#### 15.1.2.1 $(a, b, 0)$ Class

**Definition 45 ($(a, b, 0)$ Class)**

*The $(a, b, 0)$ class is a set of counting rvs with pmf $p_k$ satisfying the recursive formula*

$$\frac{p_k}{p_{k-1}} = a + \frac{b}{k}, \quad k \in \mathbb{N} \setminus \{0\}.$$

**Remark**

*An $(a, b, 0)$ distribution is determiend by the parameters $a$ and $b$.*

**Note**

*Observe that*

$$\begin{aligned}
\frac{p_1}{p_0} = a + \frac{b}{1} &\iff p_1 = p_0 \left( a + \frac{b}{1} \right) \\
\frac{p_2}{p_2} = a + \frac{b}{2} &\iff p_2 = p_1 \left( a + \frac{b}{2} \right) = p_0 \left( a + \frac{b}{1} \right) \left( a + \frac{b}{2} \right) \\
&\vdots \\
\frac{p_k}{p_{k-1}} = a + \frac{b}{k} &\iff p_k = p_0 \prod_{i=1}^{k} \left( a + \frac{b}{i} \right).
\end{aligned}$$

*Thus we see that each of the $p_k$ is completely determined by $p_0$. In other words, for the distributions of this class, if we can find $p_0$, then we can get $p_k$, even if we do not know the actual parameters of the distribution.*

*In fact, we can solve for $p_0$, if we already know what $a$ and $b$ are: we need to solve for $p_0$ in $\sum_{k=0}^{\infty} p_k = 1$. In particular, we need to solve for*

$$p_0 \sum_{k=0}^{\infty} \prod_{i=1}^{k} \left( a + \frac{b}{i} \right) = 1.$$

*Members of the $(a, b, 0)$ class* It can be shown[1] that the Poisson, Binomial, and Negative Binomial distributions are **the only** distributions that belong to this class. We have that

[1] Perhaps this can be shown using `https://www.actuaries.org/ASTIN/Colloquia/Helsinki/Papers/S7_13_Fackler.pdf`.

Table 15.1: The $(a, b, 0)$ distributions

| Distribution | $a$ | $b$ | $p_0$ |
|---|---|---|---|
| $\mathrm{Poi}(\lambda)$ | $0$ | $\lambda$ | $e^{-\lambda}$ |
| $\mathrm{Bin}(q, m)$ | $-\frac{q}{1-q}$ | $(m+1)\frac{q}{1-q}$ | $(1-q)^m$ |
| $\mathrm{NB}(\beta, r)$ | $\frac{\beta}{1+\beta}$ | $(r-1)\frac{\beta}{1+\beta}$ | $(1+\beta)^{-r}$ |

We shall prove for the case of $\mathrm{Poi}(\lambda)$.

**Exercise 15.1.2**
*Find $a$, $b$ and $p_0$ for $\mathrm{Bin}(q, m)$ and $\mathrm{NB}(\beta, r)$.*

**Proof**

By the pmf of $\mathrm{Poi}(\lambda)$, it is clear that

$$p_0 = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda}.$$

Now, since

$$p_1 = \lambda e^{-\lambda} \quad p_2 = \frac{1}{2}\lambda^2 e^{-\lambda}$$

we have the following system of equations:

$$\lambda = \frac{p_1}{p_0} = a + b$$
$$\frac{1}{2}\lambda = \frac{p_2}{p_1} = a + \frac{b}{2}$$

Thus $b = \lambda$ and $a = 0$. □

**Example 15.1.1**

Assume that the number of claims in a portfolio $N$ follows $(-0.25, 2.75, 0)$ distribution. Calculate the probability that there is at least one claim.

**Solution**

Using Table 15.1, we know that $N \sim \mathrm{Bin}(q, m)$, where

$$-\frac{q}{1-q} = -0.25 \text{ and } (m+1)\frac{q}{1-q} = 2.75,$$

which gives $q = 0.2$ and $m = 10$. Thus the desired probability is

$$P(N \geq 1) = 1 - P(N = 0) = 1 - (1 - 0.8)^{10} = 0.8926.$$

# 16 Lecture 16 Nov 6th

## 16.1 Frequency Distribution — Basic Frequency Distributioned (Continued 3)

### 16.1.1 $(a, b, n)$ Classes (Continued)

#### 16.1.1.1 $(a, b, 1)$ Class

*Motivation* There are times when the $(a, b, 0)$ class of distributions fail to give a complete characterization of certain insurance data with regards to the claim arrival process. This is especially apparent when we notice that we do not have as much freedom in fixing $P(N = 0)$. This provides us with the motivation to define the $(a, b, 1)$ class.

**Definition 46 ($(a, b, 1)$ Class)**

*The $(a, b, 1)$ class is defined as a set of counting rvs with pmf $p_k$ satisfying the recursion*

$$\frac{p_k}{p_{k-1}} = a + \frac{b}{k}, \quad k = 2, 3, \ldots.$$

**Remark**

*Notice that the formula is almost exactly the same as compared to the definition of the $(a, b, 0)$ class,* ***except*** *now we have that $k$ starts from 2 instead of 1. This means that this recursive definition will no longer have any control over $p_0$, which is what we want.*

There are two distributions from the $(a, b, 1)$ class that we shall focus on, namely

- the zero-truncated distribution; and

- the zero-modified distribution.

*Zero-Truncated Distribution* In this case, we set $p_0 = 0$, i.e. we always expect a claim.

Let $p_k$, where $k = 0, 1, 2, \ldots$, be the pmf of an $(a, b, 0)$ distribution, of which we label its rv as $N$. Then, let $p_k^T$, for $k = 0, 1, 2, \ldots$, be the pmf of the **zero-truncated distribution**, whose rv is denoted by $N^T$, with $p_0^T = 0$.

We can obtain values for each of the $p_k^T$'s, from $k = 1, 2, \ldots$, from the $p_k$'s: [1]notice that for $k = 2, 3, 4, \ldots$, we have

$$\frac{p_k^T}{p_{k-1}^T} = a + \frac{b}{k} = \frac{p_k}{p_{k-1}} \implies \frac{p_k^T}{p_k} = \frac{p_{k-1}^T}{p_{k-1}}.$$

[1] Perhaps this was implied but it certainly was not explicitly stated: the $a, b$ in the $(a, b, 0)$ can be treated exactly as the $a, b$ from the $(a, b, 0)$ class.

Observe that

$$\begin{aligned} \text{When } k = 2, \quad & \frac{p_2^T}{p_2} = \frac{p_1^T}{p_1} \\ \text{When } k = 3, \quad & \frac{p_3^T}{p_3} = \frac{p_2^T}{p_2} \\ \text{When } k = 4, \quad & \frac{p_4^T}{p_4} = \frac{p_3^T}{p_3} \\ & \vdots \end{aligned}$$

Therefore, we have that

$$\frac{p_1^T}{p_1} = \frac{p_2^T}{p_2} = \frac{p_3^T}{p_3} = \ldots =: \beta^T,$$

where we give this value a variable. Consequently, we have

$$p_k^T = \beta^T p_k.$$

Of course, we'd like to know what $\beta^T$ is. Since $p_0^T = 0$, we have that

$$1 = \sum_{k=1}^{\infty} p_k^T = \beta^T \sum_{k=1}^{\infty} p_k = \beta^T (1 - p_0)$$

and so

$$\beta^T = \frac{1}{1 - p_0}.$$

To store this information, we look to the pgf of $N^T$: we have

$$G_{N^T}(t) = \sum_{k=0}^{\infty} t^k p_k^T = \sum_{k=0}^{\infty} t^k \beta^T p_k = \frac{1}{1-p_0} \sum_{k=1}^{\infty} t^k p_k = \frac{G_N(t) - p_0}{1-p_0}.$$

**Example 16.1.1**

Let $N \sim \mathrm{NB}(\beta, r)$. Its zero-truncated version has pmf of the form

$$\begin{aligned} p_k^T = \frac{p_k}{1-p_0} &= \frac{\binom{k+r-1}{k} \left(\frac{1}{1+\beta}\right)^r \left(\frac{\beta}{1+\beta}\right)^k}{1 - \left(\frac{1}{1+\beta}\right)^r} \\ &= \frac{\Gamma(k+r)}{k!\Gamma(r)} \frac{\left(\frac{\beta}{1+\beta}\right)^k}{(1+\beta)^r - 1} \end{aligned}$$

*Zero-Modified Distribution* If we choose $P(N = 0)$ to be some value that is not any of the $p_0$'s of the distributions from the $(a, b, 0)$ class, then this distribution is called a **zero-modified distribution**.

Let $p_k$, for $k = 0, 1, 2, \ldots$, be the pmf of an $(a, b, 0)$ distribution, labelled $N$. Let $p_k^M$, for $k = 0, 1, 2, \ldots$, be the pmf of the zero-modified distribution, of which we denote by $N^M$, with $p_0^M$ chosen as described.

We can, again, express $p_k^M$ in terms of $p_k$: for $k = 1, 2, 3, \ldots$ [2],

[2] Note that in this case, the probabilities after $p_0^M$ is reliant on $p_0$; of course, since the sum of the probabilities must be 1, within the axioms of probability.

$$\frac{p_k^M}{p_{k-1}^M} = a + \frac{b}{k} = \frac{p_k}{p_{k-1}},$$

and so we have, again,

$$p_k^M = \beta^M p_k.$$

So we can solve for $\beta^M$:

$$\begin{aligned} 1 = \sum_{k=0}^{\infty} p_k^M &= p_0^M + \sum_{k=1}^{\infty} \beta^M p_k \\ \implies 1 - p_0^M &= \beta^M \sum_{k=1}^{\infty} p_k \\ \implies \beta^M &= \frac{1 - p_0^M}{1 - p_0}. \end{aligned}$$

The pgf of a zero-modified distribution is

$$\begin{aligned} G_{N^M}(t) &= \sum_{k=0}^{\infty} t^k p_k^M = p_0^M + \sum_{k=1}^{\infty} t^k \beta^M p_k = p_0^M + \beta^M (G_N(t) - p_0) \\ &= p_0^M + \frac{1-p_0^M}{1-p_0}(G_N(t) - p_0) \\ &= \frac{p_0^M - p_0^M p_0}{1-p_0} + \frac{1-p_0^M}{1-p_0} G_N(t) - \frac{p_0 - p_0^M p_0}{1-p_0} \\ &= \frac{p_0^M - p_0}{1-p_0} + \frac{1-p_0^M}{1-p_0} G_N(t) \end{aligned}$$

**Remark**

*As a consequence of the form of the pgf, if $p_0^M > p_0$, we may interpret $N_M$ as*

$$N^M = \begin{cases} 0 & w/\ prob\ \frac{p_0^M - p_0}{1-p_0} \\ N & w/\ prob\ \frac{1-p_0^M}{1-p_0} \end{cases},$$

*which is a mixture of a degenerated distribution at 0, and the original $(a, b, 0)$ distribution $N$.*

*Consequently, we can also solve for $G_{N^M}(t)$ using this notion: let $\Theta$ be the indicator-function-like distribution such that*

$$P(\Theta = \theta) = \begin{cases} \frac{p_0^M - p_0}{1-p_0} & \theta = 0 \\ \frac{1-p_0^M}{1-p_0} G_N(t) & \theta = 1 \end{cases}.$$

*Then*

$$\begin{aligned} G_{N^M}(t) &= E\left[t^{N^M}\right] = E\left[E\left[t^{N^M} \mid \Theta\right]\right] \\ &= E\left[t^{N^M} \mid \Theta = 0\right] P(\Theta = 0) + E\left[t^{N^M} \mid \Theta = 1\right] P(\Theta = 1) \\ &= P(\Theta = 0) E\left[t^0 \mid \Theta = 0\right] + P(\Theta = 1) E\left[t^N \mid \Theta = 1\right] \\ &= \frac{p_0^M - p_0}{1-p_0}(1) + \frac{1-p_0^M}{1-p_0} G_N(t), \end{aligned}$$

*as what we had.*

---

**Proposition 23 (Moments of an $(a, b, 1)$ Distribution)**

*The moments of an $(a, b, 1)$ distribution can be compited from the original $(a, b, 0)$ distribution as*

$$E\left[\left(N^M\right)^k\right] = \frac{1-p_0^M}{1-p_0} E\left[N^k\right], \quad k = 1, 2, 3, \ldots$$

*where $N^M$ is the rv modified from N from the $(a, b, 0)$ class.*

**Proof**

The derivation is straightforward:

$$\begin{aligned} E\left[\left(N^M\right)^k\right] &= \sum_{m=0}^{\infty} m^k p_m^M = \sum_{m=1}^{\infty} \frac{1-p_0^M}{1-p_0} m^k p_m \\ &= \frac{1-p_0^M}{1-p_0} \sum_{m=1}^{\infty} m^k p_m = \frac{1-p_0^M}{1-p_0} \sum_{m=0}^{\infty} m^k p_m \\ &= \frac{1-p_0^M}{1-p_0} E\left[N^k\right]. \end{aligned}$$

□

**Example 16.1.2**

Let $N \sim \text{Poi}(2)$. Find the pmf of a zero-modified version of the Poisson distribution with $p_0^M = 0.3$.

**Solution**

We are given that the pmf of $N$ is

$$p_k = \frac{e^{-2}2^k}{k!}, \quad k = 0, 1, 2, \ldots.$$

Thus

$$\begin{aligned} p_k^M &= \frac{1-p_0^M}{1-p_0} p_k = \frac{1-0.3}{1-e^{-2}} \cdot \frac{e^{-2}2^k}{k!} \\ &= \frac{0.7\left(2^k e^{-2}\right)}{k!(1-e^{-2})}, \quad k = 1, 2, 3, \ldots. \end{aligned}$$

**Note**

*It should be noted that a zero-truncated distribution is a special case of the zero-modified distribution. In fact, if $p_0^M = 0$, we get that the zero-modified distribution is exactly the zero-truncated distribution.*

In addition to the zero-modified $(a, b, 0)$ distributions, there are other members in the $(a, b, 1)$ distributions. One such example is known as the **logarithmic distribution**, which has the pmf

$$p_k = \frac{\beta^k}{k(1+\beta)^k \ln(1+\beta)}, \quad k = 1, 2, 3, \ldots,$$

with parameter $\beta > 0$. Notice that for $k = 2, 3, 4, \ldots$,

$$\begin{aligned}\frac{p_k}{p_{k-1}} &= \frac{\beta^k}{k(1+\beta)^k \ln(1+\beta)} \cdot \frac{(k-1)(1+\beta)^{k-1} \ln(1+\beta)}{\beta^{k-1}} \\ &= \frac{\beta}{1+\beta} \cdot \frac{k-1}{k} = \frac{\beta}{1+\beta} + \frac{-\frac{\beta}{1+\beta}}{k},\end{aligned}$$

where we observe that

$$a = \frac{\beta}{1+\beta} \text{ and } b = -\frac{\beta}{1+\beta}.$$

# 17 Lecture 17 Nov 13th

## 17.1 Frequency Distributions — Creating New Frequency Distributions and Effect on Frequency

### 17.1.1 Mixed Frequency Distributions

This is a concept that we have seen before.

Suppose that $N \mid \Theta = \theta$ has conditional pmf $P(N = n \mid \Theta = \theta)$ and the mixing rv $\Theta$ is either

- **discrete** with pmf $p_\Theta(\theta_i)$ for $i = 1, \ldots, m$; or
- **continuous** with pdf $f_\Theta(\theta)$.

Both the distributions of $N \mid \Theta = \theta$ and $\Theta$ are usually given. The unconditional pmf of $N$ is thus

$$P(N = n) = \begin{cases} \sum_{i=1}^{m} P(N = n \mid \Theta = \theta_i) p_\Theta(\theta_i) & \Theta \text{ is discrete} \\ \int_\Theta P(N = n \mid \Theta = \theta) f_\Theta(\theta)\, d\theta & \Theta \text{ is continuous} \end{cases}.$$

**Remark**
*Due to the context of which we work in, we shall always, perversely so, assume that $N$ is a counting rv that takes on non-negative integers.*

**Note**
*Recall Proposition 9, which gives us two concepts that are useful to us in this section: we have*

$$E\left[E\left[X \mid \Theta\right]\right] = E[X]$$
$$\mathrm{Var}(X) = \mathrm{Var}(E\left[X \mid \Theta\right]) + E\left[\mathrm{Var}(X \mid \Theta)\right].$$

#### 17.1.1.1 Mixed Poisson Distribution

**Definition 47 (Mixed Poisson Distribution)**

*If $N \mid \Lambda = \lambda$ follows $\mathrm{Poi}(\lambda)$ for some rv $\Lambda$, we say $N$ follows a* ***mixed Poisson Distribution****.*

**Example 17.1.1**

Recall Proposition 21. We have that given $N \mid \Lambda = \lambda \sim \mathrm{Poi}(\lambda)$ and $\Lambda \sim \mathrm{Gam}(\alpha, \theta)$, we have that

$$N \sim \mathrm{NB}(\theta, \alpha).$$

**Proposition 24 (Mixed Poisson Distribution has a Variance Greater than its Mean)**

*For a mixed Poisson rv $N$, we have $\mathrm{Var}(N) > E[N]$.*

**Proof**

When we say that $N$ is a mixed Poisson rv, it actually means that the rv that we have is $N \mid \Lambda = \lambda$, not $N$ alone. Now by Proposition 9, since $\Lambda = \lambda$ is the mean of the mixed Poisson rv, we have

$$\begin{aligned}\mathrm{Var}(N) &= \mathrm{Var}(E[N \mid \Lambda]) + E[\mathrm{Var}(N \mid \Lambda)] \\ &= \mathrm{Var}(\Lambda) + E[\Lambda] = \mathrm{Var}(\Lambda) + E[N].\end{aligned}$$

Since $\mathrm{Var}(\Lambda) > 0$, the result follows. □

**Example 17.1.2**

Suppose that $N \mid \Lambda = \lambda$ follows $\mathrm{Poi}(\lambda)$ and the pdf of $\Lambda$ is given by

$$f_\Lambda(\lambda) = \frac{\alpha^2}{1+\alpha}(1+\lambda)e^{-\alpha\lambda}, \quad \lambda > 0,$$

where $\alpha > 0$. Show that $N$ is the mixture of two negative binomial distributions.

**Solution**

We shall show the claim by the pgf of $N$. We have

$$\begin{aligned} G_N(t) &= E\left[t^N\right] = E\left[E\left[t^N \mid \Lambda\right]\right] = E\left[e^{\Lambda(t-1)}\right] \\ &= \int_0^\infty \frac{\alpha^2}{1+\alpha}(1+\lambda)e^{-\lambda(\alpha+1-t)}\,d\lambda \end{aligned}$$

From here, we can only proceed iff $1+\alpha-t > 0$, i.e. $t < 1+\alpha$; otherwise the integral diverges. Now, using integration by parts

$$\begin{aligned} G_N(t) = \frac{\alpha^2}{1+\alpha}\Bigg[&-\frac{1}{\alpha+1-t}e^{-\lambda(\alpha+1-t)}\Big|_0^\infty + \\ &\left[-\frac{1}{\alpha+1-t}\lambda e^{-\lambda(\alpha+1-t)}\Big|_0^\infty - \frac{1}{(\alpha+1-t)^2}e^{-\lambda(\alpha+1-t)}\Big|_0^\infty\right]\Bigg] \\ = \frac{\alpha^2}{1+\alpha}&\left[\frac{1}{\alpha+1-t}+\frac{1}{(\alpha+1-t)^2}\right] \end{aligned}$$

Now to arrive at a mixture of two negative binomial distributions, we need to know the form of the pgf for a negative binomial distribution. Note that the pgf of $\mathrm{NB}(\beta, r)$ is

$$G(t) = [1-\beta(t-1)]^{-r}.$$

Notice that

$$G_N(t) = \frac{\alpha^2}{1+\alpha}\left[[\alpha-(t-1)]^{-1} + [\alpha-(t-1)]^{-2}\right].$$

If we expand the appropriate power of $\alpha$ into the reciprocals, we can get our desired form:

$$G_N(t) = \frac{\alpha}{1+\alpha}\left[1-\frac{1}{\alpha}(t-1)\right]^{-1} + \frac{1}{1+\alpha}\left[1-\frac{1}{\alpha}(t-1)\right]^{-2}.$$

It is clear that

$$\frac{\alpha}{1+\alpha}+\frac{1}{1+\alpha}=1,$$

and so we have obtained our desired result; that is $N$ is a weighted mixture of two negative binomial distributions. In particular,

$$N = \begin{cases} X_1 \sim \mathrm{NB}\left(\frac{1}{\alpha}, 1\right) & \text{w/ prob } \frac{\alpha}{1+\alpha} \\ X_2 \sim \mathrm{NB}\left(\frac{1}{\alpha}, 2\right) & \text{w/ prob } \frac{1}{1+\alpha} \end{cases}.$$

We can make a similar derivation of mixed distributions for Binomial and Negative Binomial.

### 17.1.2 Compound Frequency Distributions

**Definition 48 (Compound Frequency Distribution)**

*For two counting rvs N and M, let*

$$S = \sum_{i=1}^{N} M_i = \begin{cases} M_1 + \ldots + M_N & N \geq 1 \\ 0 & N = 0 \end{cases},$$

*where $\{M_i\}_{i=1}^{\infty}$ is a sequence of iid rv's distributed as M, and are independent of N. We call S a* ***compound rv****, N the* ***primal distribution****, and M the* ***secondary distribution****.*

**Remark**

- *Compounding two counting rvs is also an approach to create new frequency distributions.*
- *S is called* ***compound Poisson, Binomial, or Negative Binomial*** *if N is a Poisson, a Binomial, or a Negative Binomial rv, respectively.*

**Example 17.1.3 (Interpretation)**

In an insurance context, compound rvs arise rather naturally. E.g. in the auto insurance context, we could have

- $N$ represents the number of accidents;
- $M_i$ represents the number of claims generated by the $i^{\text{th}}$ accident;
- And so in this case $S$ stands for the total number of claims for a portfolio of auto insurance policies over a given time period.

**Proposition 25 (Mean and Variance of the Compound RV)**

*For a compound rv $S = \sum_{i=1}^{N} M_i$, where $M_i \sim M$, we have*

$$E[S] = E[N]E[M]$$
$$\text{Var}(S) = \text{Var}(N)E[M]^2 + E[N]\,\text{Var}(M).$$

**Proof**

Note that the definition of $S$ relies on $N$ first, since

$$S = \begin{cases} M_1 + \ldots + M_N & N \geq 1 \\ 0 & N = 0 \end{cases},$$

so we shall go down of the route of conditioning $S$ by $N$. Observe that

$$E[S \mid N] = E\left[\sum_{i=1}^{N} M_i \mid N\right] \overset{(*)}{=} \sum_{i=1}^{N} E[M_i \mid N] \overset{(**)}{=} \sum_{i=1}^{N} E[M] = NE[M]$$

where $(*)$ is by the linearity of the expectation, and $(**)$ is by $M_i \perp N$ for each $i$ and that $M_i \sim M$. The variance of $S$ conditioned on $N$ is

$$\mathrm{Var}(S \mid N) = \mathrm{Var}\left(\sum_{i=1}^{N} M_i \mid N\right) = \sum_{i=1}^{N} \mathrm{Var}(M) = N\,\mathrm{Var}(M)$$

mostly for the same reason as for the expectation, but the 2nd equality involves independence of the $M_i$'s (otherwise, we would be left with a bunch of covariances).

Then, using Proposition 9, we have

$$E[S] = E[E[S \mid N]] = E[NE[M]] = E[N]E[M]$$

and

$$\begin{aligned} \mathrm{Var}(S) &= \mathrm{Var}(E[S \mid N]) + E[\mathrm{Var}(S \mid N)] \\ &= \mathrm{Var}(NE[M]) + E[N\,\mathrm{Var}(M)] \\ &= \mathrm{Var}(N)E[M]^2 + E[N]\,\mathrm{Var}(M) \end{aligned}$$

as required. □

**Note (Notation)**

*Hereafter, we shall use the following notations: notice that each M, N, and S are counting rvs, i.e. they are discrete and are non-negative integers, so let*

- *$p_k$ represent the pmf of N, the primal distribution;*
- *$f_k$ represent the pmf of M, the secondary distribution; and*
- *$g_k$ represent the pmf of S, the compound rv.*

---

In the next lecture, we shall look into how to compuete $g_k$. Namely, we have the following three methods:

- pgf method;
- pmf method; and
- Panjer's recursion.

# 18 Lecture 18 Nov 15th

## 18.1 Frequency Distributions — Creating New Frequency Distributions and Effect on Frequency (Continued)

### 18.1.1 Compound Frequency Distributions (Continued)

#### 18.1.1.1 PGF Method

**Proposition 26 (PGF Method for Compound RVs)**

*For a compound rv $S = \sum_{i=1}^{N} M_i$, we have*

$$G_S(t) = G_N(G_M(t)),$$

*where $G_S, G_N, G_M$ are pgfs of $S, N, M$, respectively.*

**Proof**

Given the definition of $S$, we have

$$\begin{aligned} G_S(t) = E\left[t^S\right] &= E\left[E\left[t^{M_1+M_2+\ldots+M_N} \mid N\right]\right] = E\left[\prod_{i=1}^{N} E\left[t^{M_i} \mid N\right]\right] \\ &= E\left[\left(E\left[t^M\right]\right)^N\right] = E\left[G_M(t)^N\right] = G_N(G_M(t)). \end{aligned}$$

**Note**

*From Proposition 26, the pmf of $g_k$ can be computed using*

$$g_k = \frac{1}{k!} G_S^{(k)}(0).$$

*E.g.*

- $g_0 = G_S(0) = G_N(G_M(0)) = G_N(f_0)$
- $g_1 = G_S'(0) = G_M'(0)G_N'(G_M(0)) = f_1 G_N(f_0)$

*However, this method is inefficient.*

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#### 18.1.1.2 PMF Method

We shall develop, perhaps, a more efficient way of getting $g_k$: For $k = 0, 1, 2, \ldots$, we have

$$\begin{aligned} g_k = P(S = k) &= \sum_{n=0}^{\infty} P(S = k \mid N = n) P(N = n) \\ &= \sum_{n=0}^{\infty} P(M_1 + \ldots + M_n = k) p_n \\ &= \sum_{n=0}^{\infty} f_k^{*n} p_n, \end{aligned}$$

where we let $f_k^{*n}$ be the **$n$-fold convolution** of $f_k$ defined as

$$f_k^{*n} := P(M_1 + \ldots + M_n = k).$$

Note that when $k = 0$, we have

$$g_0 = \sum_{n=0}^{\infty} f_0^{*n} p_n = f_0^{*0} p_0 + \sum_{n=1}^{\infty} f_0^{*n} p_n.$$

Note that

$$f_0^{*0} = P(0 = 0) = 1,$$

thus

$$\begin{aligned} g_0 &= p_0 + \sum_{n=1}^{\infty} f_0^{*n} p_n \\ &= p_0 + \sum_{n=1}^{\infty} P(M_1 + \ldots + M_n = 0) p_n \\ &= p_0 + \sum_{n=1}^{\infty} p_n \prod_{i=1}^{n} P(M = 0) \\ &= p_0 + \sum_{n=1}^{\infty} p_n \prod_{i=1}^{n} f_0 \\ &= p_0 + \sum_{n=1}^{\infty} f_0^n p_n. \end{aligned}$$

Now in general,

$$g_k = \sum_{n=0}^{\infty} f_k^{*n} p_n = f_k^{*n} p_0 + \sum_{n=1}^{\infty} f_k^{*n} p_n.$$

Observe that

$$f_k^{*0} = P(0 = k) = 0.$$

Thus

$$g_k = \sum_{n=1}^{\infty} f_k^{*n} p_n,$$

for $k \geq 1$.

We are still short of an important information: what exactly is $f_k^{*n}$? Fix $n \in \mathbb{N}$. Observe that we only need to look for the value of $n-1$ of the $M_i$'s, since the sum of the $M_i$'s must equal to some $k$. Then by using the methods from conditional probability

$$\begin{aligned} f_k^{*n} &= P(M_1 + \ldots + M_n = k) \\ &= \sum_{j=0}^{k} P(M_1 + \ldots + M_n = k \mid M_n = j) P(M_n = j) \\ &= \sum_{j=0}^{k} P(M_1 + \ldots + M_{n-1} = k - j) f_j \\ &= \sum_{j=0}^{k} f_{k-j}^{*(n-1)} f_j, \end{aligned}$$

and we observe that we have a recursive definition. Fortunately, this recursive definition has a start where we can work with: note that $f_k^{*1} = P(M_1 = k) = f_k$.

However, for large values of $n$, this method becomes very cumbersome.

**Example 18.1.1**

Suppose that $p_0 = 0.4$, $p_1 = 0.4$, and $p_2 = 0.2$. Also, we have $f_0 = 0.5$, $f_1 = 0.3$, and $f_2 = 0.2$. Find the pmf of $S$.

✎ **Solution**

First, notice that since $N, M$ can take values $0, 1, 2$. Thus $S$ can take values $0, 1, 2, 3, 4$. Observe that

$$\begin{aligned} g_0 &= p_0 + \sum_{n=1}^{2} f_0^n p_n = 0.4 + 0.5(0.4) + 0.5^2(0.2) = 0.65 \\ g_k &= \sum_{n=1}^{2} f_k^{*n} p_n = 0.4 f_k^{*1} + 0.2 f_k^{*2} \\ &= 0.4 f_k + 0.2 f_k^{*2}, \quad \text{for } k = 1, 2, 3, 4. \end{aligned}$$

It remains to solve for $f_k^{*2}$, for $k = 1, 2, 3, 4$.

$$\begin{aligned} f_1^{*2} &= P(M_1 + M_2 = 1) = P(M_1 = 1, M_2 = 0) + P(M_1 = 0, M_2 = 1) \\ &= 2 f_0 f_1 = 0.3 \\ f_2^{*2} &= P(M_1 + M_2 = 2) \\ &= f_0 f_2 + f_1 f_1 + f_2 f_0 = 0.29 \\ f_3^{*2} &= P(M_1 + M_2 = 3) \\ &= f_1 f_2 + f_2 f_1 = 0.12 \\ f_4^{*2} &= P(M_1 + M_2 = 4) \\ &= f_2 f_2 = 0.04. \end{aligned}$$

We can then obtain $g_k$ for $k = 1, 2, 3, 4$ from here.

**Remark**

*We observe that in the previous example, the grand method of which we derived is rather cumbersome to work with, even if we just have a maximum of* 4*.*

Now recall Proposition 26. We shall try using this. To that end, we first need to get $G_N(t)$ and $G_M(t)$, which is not difficult:

$$\begin{aligned} G_N(t) &= 0.4 + 0.4t + 0.2t^2 \\ G_M(t) &= 0.5 + 0.3t + 0.2t^2. \end{aligned}$$

Then

### 18.1.1.3 *Panjer's Recursion*

**Theorem 27 (Panjer's Recursion for $(a, b, 0)$ class)**

*For an $(a, b, 0)$ distribution $N$, the pmf of $S$ can be calculated, recursively,* *by*

$$g_k = \frac{1}{1 - af_0} \sum_{i=1}^{k} \left( a + \frac{ib}{k} \right) f_i g_{k-1}, \quad k = 1, 2, \ldots,$$

*where the starting point of the recursion is*

$$g_0 = G_N(f_0).$$

**Proof**

**Proof to be added later.** Reference to add `https://www.casact.org/library/astin/vol12no1/22.pdf`

**Example 18.1.2**

If $N$ is a Poisson rv, or equivalently, if $S$ is a compound Poisson rv, then

$$g_k = \sum_{i=1}^{k} \frac{ib}{k} f_i g_{k-1}, \quad k = 1, 2, \ldots.$$

**Theorem 28 (Panjer's Recursion for $(a, b, 1)$ Class)**

*For an $(a, b, 1)$ distribution $N$, the pmf of $S$ can be calculated recursively by*

$$g_k = \frac{p_1 - (a+b)p_0}{1 - af_0} f_k + \frac{1}{1 - af_0} \sum_{i=1}^{k} \left( a + \frac{ib}{k} \right) f_i g_{k-i}, \quad k = 1, 2, \ldots,$$

*where*

$$g_0 = G_N(f_0).$$

# 19 Lecture 19 Nov 20th

## 19.1 Frequency Distributions — Creating New Frequency Distributions and Effect on Frequency (Continued 2)

### 19.1.1 Effect on Frequency

Let $N$ denote the number of claims generated by a portfolio of insurance policies. We shall analize the effects on $N$ from two kinds of adjustments on the portfolio:

- exposure adjustment;
- policy adjustment.

#### 19.1.1.1 Exposure Adjustment

Suppose that the total number of claims of an insurance portfolio consisting $k$ policies in the current year is modelled by a counting number $N$ with pgf $G_N(t)$, and suppose that we do not have the model for the number of claims of each policy.

QUESTION: In the following year, if the number of policies changes to some $k^*$, and we let $N^*$ be the total number of claims in the new year, what is the relationship between the pgfs of $N$ and $N^*$?

**Example 19.1.1**

Let $N_i$ be the number of claims produced by the $i^{\text{th}}$ policy. Then we have

$$N = \sum_{i=1}^{k} N_i \text{ and } N^* = \sum_{i=1}^{k^*} N_i.$$

Also, suppose that allof the $N_i$'s are iid. Then

$$\begin{aligned} G_N(t) &= E\left[t^{N_1+N_2+\ldots+N_k}\right] = \left(E\left[t^{N_1}\right]\right)^k \\ G_{N^*}(t) &= E\left[t^{N_1+N_2+\ldots+N_{k^*}}\right] = \left(E\left[t^{N_1}\right]\right)^{k^*} \\ &= \left[\left(E\left[t^{N_1}\right]\right)^k\right]^{\frac{k^*}{k}} = \left[G_N(t)\right]^{\frac{k^*}{k}}. \end{aligned}$$

However, it may not be the case where $\frac{k^*}{k}$ is a ratio such that $[G_N(t)]^{\frac{k^*}{k}}$ is still a pgf.

**Definition 49 (Infinitely Divisible)**

*A discrete distribution with pgf $G(t)$ is said to be* ***infinitely divisible*** *if for all $k = 1, 2, \ldots$, the function $[G(t)]^{\frac{1}{k}}$ is the pgf of some rv.*

**Remark (Namesake)**
*The name comes from the idea that the rv can be "infinitely divided" into k-many iid rvs, for any $k \in \mathbb{N}$, e.g.*

$$X = Y_1 + \ldots + Y_k.$$

**Remark**

1. *If $G(t)^{\frac{1}{k}}$ is a proper pgf, then so is $G(t)^{\frac{n}{k}}$.*
2. *If $G(t)$ is a proper pgf, then so is $G(t)^n$.*

*The 2nd remark is true since if we let $X$ have the pgf $G(t)$, then $G(t)^n$ is the pgf of n-many independent copies of $X$ (i.e. we have an iid sequence $\{X_i\}_{i=1}^n$ with $X_i \sim X$).*

**Example 19.1.2**

The Poisson and negative binomial distributions are infinitely divisible, but the binomial distribution is not.

**Solution**
Let $N \sim \text{Poi}(\lambda)$. Then for any $k \in \mathbb{N} \setminus \{0\}$, we have

$$G_N(t) = e^{\lambda(t-1)} \implies [G_N(t)]^{\frac{1}{k}} = e^{\frac{\lambda}{k}(t-1)}$$

which is a pgf of $\text{Poi}\left(\frac{\lambda}{k}\right)$.

If $N \sim \text{NB}(\beta, r)$, then for any $k \in \mathbb{N} \setminus \{0\}$, we have

$$G_N(t) = (1+\beta-\beta t)^{-r} \implies [G_N(t)]^{\frac{1}{k}} = (1+\beta-\beta t)^{-\frac{r}{k}}$$

which is the pgf of $\text{NB}\left(\beta, \frac{r}{k}\right)$.

For $N \sim \text{Bin}(q, m)$, notice that for $k \in \mathbb{N} \setminus \{0\}$, we have

$$G_N(t) = (1-q+qt)^m \implies [G_N(t)]^{\frac{1}{k}} = (1-q+qt)^{\frac{m}{k}}.$$

However, $\frac{m}{k}$ may not be an integer. For example, if $k = 2m$, then we have

$$G(t) = [G_N(t)]^{\frac{1}{m}} = (1-q+qt)^{\frac{1}{2}}.$$

Note that $G(t)$ is a power series, since it is **differentiable infinitely**[1]. Then in particular, notice that

$$\begin{aligned} \frac{d^2}{dt^2} [G_N(t)]^{\frac{1}{2m}} \Big|_{t=0} &= -\frac{1}{4} q^2 (1-q+qt)^{-\frac{3}{2}} \Big|_{t=0} \\ &= -\frac{1}{4} q^2 (1-q)^{-\frac{3}{2}} < 0. \end{aligned}$$

[1] In comparison, the pgf of a binomial distribution is differentiable $m$-times, where $m$ is the counting parameter in the binomial distribution.

**Example 19.1.3 (Infinite Divisibility of Compound RVs)**

Recall that for a compound rv $S = \sum_{i=1}^{N} M_i$, we have that

$$G_S(t) = G_N(G_M(t)).$$

Then if **$N$ is infinitely divisible**, then so is $S$, since

$$[G_S(t)]^{\frac{1}{k}} = [G_N(G_M(t))]^{\frac{1}{k}}.$$

**Remark**

*As a consequence of the last two examples, we have that **compound Poisson** and **compound negative binomial** distributions are infinitely divisible.*

#### 19.1.1.2 *Effect of Policy Adjustments*

Let $X$ be the ground-up loss and $N$ be the number of losses from a portfolio of risks. The amount paid (either $Y_L$ or $Y_P$)[2] is an amount modified from $X$ using policy adjustments.

[2] See

There are two common ways to define the model reflecting the aggregate payment:

- the aggregate payment on a **per-loss basis**;

- the aggregate payment on a **per-payment basis**.

*Aggregate Payment on a per-loss basis* Let $N_L$ be the number of payments paid on a per-loss basis. Then the payment of every loss is included, and so we have

$$N_L = N.$$

Thus the **aggregate payment on a per-loss basis** is

$$S = \sum_{i=1}^{N_L} Y_{L,i},$$

where $Y_L, i$ denotes the amount paid for the $i^{\text{th}}$ loss.

*Aggregate Payment on a per-payment basis* Let $N_P$ be the number of payments paid on a per-payment basis. Then only the non-zero payments are counted, and so we have

$$N_P = \sum_{i=1}^{N_L} \mathbb{1}_{\{Y_{L,i} > 0\}}.$$

Thus the **aggregate payment on a per-payment basis** is

$$S = \sum_{i=1}^{N_P} Y_{P,i},$$

where $Y_{P,i}$ denotes the amount paid for the $i^{\text{th}}$ non-zero payment.

Notice that if we let $I_i = \mathbb{1}_{\{Y_{L,i} > 0\}}$ and let $P(Y_L > 0) = \alpha$, then the pgf of $N_P$ is given by (using Proposition 26)

$$G_{N_P}(t) = G_{N_L}(G_{I_1}(t)) = G_{N_L}(1 - \alpha + \alpha t)$$

since $I_i \sim \text{Bernoulli}(\alpha)$.

# 20 Lecture 20 Nov 22nd

## 20.1 Frequency Distributions — Creating New Frequency Distributions and Effect on Frequency Distribution (Continued 3)

### 20.1.1 Effect on Frequency (Continued)

#### 20.1.1.1 Effect of Policy Adjustments (Continued)

**Example 20.1.1**

Assume that $P(Y_L > 0) = \alpha$. Find the corresponding distribution of $N_P$ given the following distributions of $N_L$.

1. $N_L \sim \text{Poi}(\lambda)$;
2. $N_L \sim \text{Bin}(m, q)$;
3. $N_L \sim \text{NB}(\beta, r)$.

**Solution**

1. Given $N_L \sim \text{Poi}(\lambda)$, we have that the pgf of $N_P$ is

$$\begin{aligned} G_{N_P}(t) &= G_{N_L}(G_{\mathbb{1}_{\{Y_{L,i}>0\}}}(t)) = G_N(1 - \alpha + \alpha t) \\ &= e^{\lambda(1-\alpha+\alpha t-1)} = e^{\lambda\alpha(t-1)} \end{aligned}$$

   which is the pgf of $\text{Poi}(\lambda\alpha)$. Thus $N_P \sim \text{Poi}(\lambda\alpha)$.

2. Given $N_L \sim \text{Bin}(m, q)$, we have that the pgf of $N_P$ is

$$\begin{aligned} G_{N_P}(t) &= G_N(1 - \alpha + \alpha t) = (1 - q + (1 - \alpha + \alpha t)q)^m \\ &= (1 - \alpha q + t\alpha q)^m \end{aligned}$$

   which is the pgf of $\text{Bin}(m, \alpha q)$. Thus $N_P \sim \text{Bin}(m, \alpha q)$.

3. Given $N_L \sim \text{NB}(\beta, r)$, we have that the pgf of $N_P$ is

$$G_{N_P}(t) = [1 - \beta(1 - \alpha + \alpha t - 1)]^{-r} = [1 - \beta\alpha(t-1)]^{-r}$$

which is the pgf of $\text{NB}(\beta\alpha, r)$. Thus $N_P \sim \text{NB}(\beta\alpha, r)$.

**Remark**

*Also, to solve for $N_P$ for when $N_L \sim \text{Poi}(\alpha\lambda)$, recall Example 14.1.1.*

## 20.2 Aggregate Risk Models

Textbook reference: Sections 9.1 - 9.7

We shall now look back at some of the things that we introduced throughout the course and put them together.

### 20.2.1 Indiviual Risk Model Revisited

Recall the individual risk model. Consider a portfolio consisting of $n$ insurance policies. Let $Z_i$ denote the amount paid for the $i^{\text{th}}$ policy. The individual risk model for the aggregate amount paid is given by

$$S = \sum_{i=1}^{n} Z_i.$$

While we are interested in the distribution of $S$, it is usually difficult to obtain an analytical form for the distribution.

But if $n$ is large, and $\{Z_i\}_{i=1}^{n}$ is an iid sequence, we can apply the **normal approximation** (by the **central limit theorem**) to solve for $S$: in particular, as $n \to \infty$, we have

$$\frac{S - E[S]}{\sqrt{\text{Var}(S)}} \xrightarrow{d} \text{N}(0, 1),$$

where $\xrightarrow{d}$ means a **convergence in distribution**[1]

[1] See notes from STAT330.

**Example 20.2.1**

Consider a portfolio of 100 iid insurance policies. Each policy is applied to a ground-up loss that follows an exponential distirbution with mean 50. An ordinary deductible $d = 20$ is applied to each policy and the insurer charges premium as 1.1 multiple of the expected payment amount. Find the probability that the insurer will have a negative profit with this portfolio by using the normal approximation.

**Solution**

The amount paid for the $i^{\text{th}}$ policy is $Z_i = [X_i - 20]_+$, where $X_i \sim \text{Exp}(50)$. We want to find

$$P(1.1E[S] - S < 0) = P(S > 1.1E[S]).$$

Since $n$ is large enough (we suppose so), we can use normal approximation:

$$P(S > 1.1E[S]) = P\left(\frac{S - E[S]}{\sqrt{\text{Var}(S)}} > \frac{1.1E[S] - E[S]}{\sqrt{\text{Var}(S)}}\right)$$

where $\frac{S-E[S]}{\sqrt{\text{Var}(S)}} \xrightarrow{d} \text{N}(0,1)$. Thus we need to solve for

$$\begin{aligned} E[S] &= E\left[\sum_{i=1}^{100} Z_i\right] = 100E\left[[X-20]_+\right] \\ \text{Var}(S) &= \text{Var}\left(\sum_{i=1}^{100} Z_i\right) = 100\,\text{Var}(Z_i) = 100\,\text{Var}([X-20]_+). \end{aligned}$$

We have

$$\begin{aligned} E[[X-20]_+] &= \int_{20}^{\infty} e^{-\frac{1}{50}x}\,dx = -50e^{-\frac{x}{50}}\Big|_{20}^{\infty} = 33.516 \\ E[[X-20]_+^2] &= \int_{20}^{\infty} 2(x-20)e^{-\frac{x}{50}}\,dx \\ &= -2000e^{-0.4} + 2\left[1000e^{-0.4} + 50[50e^{-0.4}]\right] \\ &= 3351.600 \end{aligned}$$

Thus

$$E[S] = 3351.6 \text{ and } \text{Var}(S) = 100(3351.6 - 33.516^2) = 222827.7744.$$

Therefore

$$\begin{aligned} P(X > 1.1E[S]) &= P\left(\frac{S-E[S]}{\sqrt{\text{Var}(S)}} > \frac{0.1E[S]}{\sqrt{\text{Var}(S)}}\right) \\ &= P\left(\frac{S-E[S]}{\sqrt{\text{Var}(S)}} > \frac{335.16}{\sqrt{222827.7744}}\right) \\ &\approx P\left(\frac{S-E[S]}{\sqrt{\text{Var}(S)}} > 0.71\right) \\ &\approx 1 - P\left(Z \leq 0.71\right) = 0.2389 \end{aligned}$$

where $Z \sim \text{N}(0,1)$.

### 20.2.2 Collective Risk Model Revisited

Recall the collective risk model: we define the aggregate loss as a random sum of severity amounts, i.e.

$$S = \sum_{i=1}^{N} X_i,$$

where

- $N$ is a counting rv,
- $\{X_i\}_{i \geq 1}$ is a sequence of iid severity rvs,
- and we follow the convention that $S = 0$ if $N = 0$.

We are, again, interested in the aggregate amount paid by the insurer. The collective risk model for the aggregate amount paid can be expressed in two forms, as mentioned in Section 19.1.

We first study the distribution of $S$ when the severity distribution $X$ is a **discrete rv** with pmf

$$f_X(x) = P(X = x), \quad x = 0, 1, 2, \ldots.$$

We have several ways to solve for $S$:

- Notice that $S$ is also a discrete rv with pmf $f_X(x) = P(S = x)$ for $x = 0, 1, 2, \ldots$, and pgf of $S$ is

  $$G_S(t) = G_N(G_X(t))$$

  as discussed in Proposition 26.

- $S$ is a compound rv with primary rv $N$ and secondary rv $X$, so we can use methods from Section 18.1.1.

For example, if $N$ is a member of the $(a, b, 0)$ or $(a, b, 1)$ class, we can use Panjer's Recursion to find the pmf of $S$.

It may be the case that $N$ itself is yet another compound rv with primary rv $N_1$ and secondary rv $N_2$, and in this case the pgf of $S$ would be

$$G_S(t) = G_N(G_X(t)) = G_{N_1}(G_{N_2}(G_X(t))) = G_{N_1}(G_Z(t)),$$

where we let $Z$ be the compound rv with primary rv $N_2$ and secondary rv $X$.

Again, if $N_1$ and $N_2$ are **either members of** $(a, b, 0)$ **or** $(a, b, 1)$, then we can apply Panjer's Recursion in the following manner:

Step 1 Apply Panjer's Recursion on $Z$.

Step 2 Apply Panjer's Recursion on $S$.

*General Discrete Severity Distribution* Suppose the severity rv $X$ has a pmf

$$f_X(x) = P(X = xh), \quad x = 0, 1, 2, \ldots,$$

where $h > 0$ is some constant[2]. In this case, let

$$\tilde{S} = \frac{S}{h} = \sum_{i=1}^{N} \tilde{X}_i,$$

where $\{\tilde{X}_i\}_{i \geq 1}$ is a sequence of iid copies such that $\tilde{X}_i = \frac{X_i}{h}$.

[2] Here, $h$ acts as a scale for the values that the rv can take on, allowing us to take non-integer values in a discrete manner.

# 21 Lecture 21 Nov 27th

## 21.1 Aggregate Risk Models Revisited

### 21.1.1 Collective Risk Model Revisited (Continued)

*Continuous Severity Distribution* Suppose that $X$ is a continuous severity rv with pdf $f_X$. Since we usually have that $P(N = 0) > 0$, $S$ is a **mixture with probability mass at** $0$.

**Proposition 29 (MGF of Aggregate Loss of A Continuous Severity Distribution)**

*The mgf of aggregate loss $S$ is given by*

$$M_S(t) = G_N(M_X(t)),$$

*where $G_N$ is the pgf of frequency rv $N$, and $M_X$ is the mgf of severity rv $X$.*

**Proof**

By conditioning on $N$, we have

$$\begin{aligned} M_S(t) &= E\left[e^{tS}\right] = E\left[E\left[e^{t(X_1+\ldots+X_N)} \mid N\right]\right] \\ &= E\left[E\left[e^{tX_1} \mid N\right] \ldots E\left[e^{tX_N} \mid N\right]\right] \\ &= E\left[E\left[e^{tX}\right]^N\right] = E\left[M_X(t)^N\right] = G_N(M_X(t)). \end{aligned}$$

□

The distribution of $S$ is often closely related to the **Erlang distribution**, which is a special case of $\mathrm{Gam}(\alpha, \theta)$ with $\alpha$ as a **positive integer**.

**Definition 50 (Erlang Distribution)**

*A crv $X$ is said to follow an **Erlang Distribution**, dentoed as $X \sim \mathrm{Erlang}(n, \theta)$, if its pdf is*

$$f_X(x) = \frac{\theta^{-n} x^{n-1} e^{-\frac{x}{\theta}}}{\Gamma(n)}.$$

**Remark**

*We usually do not have a nice closed form cdf of a Gamma distribution, except when $\alpha$ is an integer. Thus $X \sim \mathrm{Erlang}(n, \theta)$ has the cdf*

$$F_X(x) = 1 - \int_x^\infty \frac{\theta^{-n} y^{n-1} e^{-\frac{y}{\theta}}}{(n-1)!} \, dy = 1 - \sum_{k=0}^{n-1} \frac{\left(\frac{x}{\theta}\right)^k e^{-\frac{x}{\theta}}}{k!}.$$

*Also, from the mgf of the Gamma distribution, we have that the mgf of the $\mathrm{Erlang}(n, \theta)$ is*

$$M(t) = (1 - \theta t)^{-n}.$$

**Example 21.1.1**

Suppose that the frequency rv $N$ has pmf $\{p_k\}_{k=0}^\infty$, and the severity rv $X \sim \mathrm{Exp}(\theta)$. What is the distribution of the aggregate loss $S$?

**Solution**

By Proposition 29, since $G_N(t) = \sum_{k=0}^\infty t^k p_k$ and $M_X(t) = (1 - \theta t)^{-1}$, we have

$$\begin{aligned} M_S(t) = G_N(M_X(t)) &= \sum_{k=0}^\infty M_X(t)^k p_k = \sum_{k=0}^\infty (1-\theta t)^{-k} p_k \\ &= p_0 + \sum_{k=1}^\infty (1 - \theta t)^{-k} p_k. \end{aligned}$$

Thus we observe that $S$ is a mixture of 0 and $\mathrm{Erlang}(k, \theta)$, for $k = 1, 2, \ldots$.

**Note**

*Note that since $S$ is a mixture, we can obtain its pf and cdf by taking* *appropriate weights on each of the distributions in its mixture.*

---

*Normal Approximation of Collective Risk Model* We can also use the normal approximation method for a collective risk model, in particular, given $S = \sum_{i=1}^{N} X_i$ for crvs $X_i$'s, we have

$$P(S \leq x) \approx P\left( Z \leq \frac{x - E[S]}{\sqrt{\mathrm{Var}(S)}} \right),$$

where $Z \sim \mathrm{N}(0, 1)$. If the primary distribution $N$ is from the $(a, b, 0)$ class, then the conditions required for applying the normal approximation is

- large $\lambda$;
- large $m$; and
- large $r$

for $N \sim \mathrm{Poi}(\lambda)$, $N \sim \mathrm{Bin}(m, q)$, and $N \sim \mathrm{NB}(r, q)$, respectively.

**Example 21.1.2**

Let $X \sim \mathrm{Exp}(2)$, and $N \sim \mathrm{Poi}(50)$. Use the normal approximation to find the 95% quantile of $S$.

**Solution**
Let $X_i \sim X$, for $i = 1, 2, \ldots, N$. Then by Proposition 25, we have

$$E[S] = E\left[\sum_{i=1}^{N} X_i\right] = E[N]E[X] = 100,$$

and

$$\begin{aligned}\mathrm{Var}(S) = \mathrm{Var}\left(\sum_{i=1}^{N} X_i\right) &= E\left[\mathrm{Var}\left(\sum_{i=1}^{N} X_i \mid N\right)\right] + \mathrm{Var}\left(E\left[\sum_{i=1}^{N} X_i \mid N\right]\right) \\ &= E[N]\,\mathrm{Var}(X) + \mathrm{Var}(N)E[X]^2 = 400.\end{aligned}$$

The 95% quantile, $\pi_{0.95}$, is such that

$$P(S < \pi_{0.95}) \leq 0.95 \leq P(S \leq \pi_{0.95}).$$

Then

$$\begin{aligned} 0.95 = F_S(\pi_{0.95}) &= P(S \leq \pi_{0.95}) \\ &= P\left( \frac{S - E[S]}{\sqrt{\mathrm{Var}(S)}} \leq \frac{\pi_{0.95} - E[S]}{\sqrt{\mathrm{Var}(S)}} \right) \\ &\approx P\left( Z \leq \frac{\pi_{0.95} - 100}{\sqrt{400}} \right) = \Phi\left( \frac{\pi_{0.95} - 100}{20} \right). \end{aligned}$$

Thus

$$\frac{\pi_{0.95} - 100}{20} = \Phi^{-1}(0.95) = 1.645$$

and so

$$\pi_{0.95} = 121.29.$$

#### 21.1.1.1 *Discretization Method*

The normal approximation has limitations: e.g. how large is large for each of the required parameters.

---

**Definition 51 (Discretization Method)**

*The* ***discretization method*** *is to approximate the distribution of a continuous (or mixed) severity rv $X$ by the distribution of a discrete severity rv $\hat{X}$, which has mass points at $\{0, h, 2h, \ldots\}$. The pmf of $\hat{X}$ is*

$$\begin{aligned} P\left(\hat{X} = 0\right) &= P\left(0 \leq X \leq \frac{h}{2}\right) \\ P\left(\hat{X} = kh\right) &= P\left(kh - \frac{h}{2} < X \leq kh + \frac{h}{2}\right) \end{aligned}$$

*for $k = 1, 2, \ldots$.*

---

**Remark**

*Recall that the aggregate payment can be expressed as*

$$S = \sum_{i=1}^{N_L} Y_{L,i} \text{ or } S = \sum_{i=1}^{N_P} Y_{P,i}.$$

*Since the amount paid per payment, $Y_P$, is usually continuous (especially so at 0), we usually apply the discretization method to $Y_P$, and then obtain an approximation to the distribution of $S$.*

# 22 Lecture 22 Nov 29th

## 22.1 Aggregate Risk Models Revisited (Continued)

### 22.1.1 Collective Risk Model Revisited (Continued 2)

#### 22.1.1.1 Discretization Method (Continued)

**Example 22.1.1**

Assume that the number of losses $N \sim \text{Poi}(3)$. The ground-up loss $X \sim \text{Pareto}(4, 10)$. An individual loss limit of 15 and an ordinary deductible of 6 are applied to each loss. Determine the distribution of the discretized aggregate payment with $h = 2.5$.

The goal is to derive the distribution of a discretized aggregate payment $\hat{S} = \sum_{i=1}^{N_P} \hat{Y}_{P,i}$. To that end, we need to figure out the secondary distribution $\hat{Y}_{P,i}$ and the primary distribution $N_P$. For each of the $\hat{Y}_{P,i}$'s, we need to use the discretized method:

$$P\left(\hat{Y}_{P,i} = 0\right) = P\left(0 \leq Y_{P,i} < \frac{h}{2}\right)$$

$$P\left(\hat{Y}_{P,i} = k\right) = P\left(\frac{(2k-1)h}{2} \leq Y_{P,i} < \frac{(2k+1)h}{2}\right)$$

We know what $Y_{P,i}$ is from $Y_{L,i}$. To find out what $N_P$ is, we are given that $N_L = N \sim \text{Poi}(3)$, and so we may use the relation

$$G_{N_p}(t) = G_{N_L}(1 - \alpha + \alpha t),$$

where $\alpha$ is the probability that a loss occurs.

**Solution**

Let $Y_L = [(X \wedge 15) - 6]_+$, let $\{X_i\}$ be such that $X_i \sim X$, and $\{Y_{L,i}\}$ be such that $Y_{L,i} = [(X_i \wedge 15) - 6]_+$ for each $i$. It follows that by this construction, we have $Y_{L,i} \sim Y_L$. Now, since $X \sim \text{Pareto}(4, 10)$, the survival function of $X$ is

$$\bar{F}_X(x) = \left(\frac{\theta}{x+\theta}\right)^{\alpha} = \left(\frac{10}{x+10}\right)^4.$$

It follows that the survival function of $Y_L$ is

$$\bar{F}_{Y_L}(y) = \begin{cases} 1 & y < 0 \\ \bar{F}_X(y+6) & 0 \leq y < 9 \\ 0 & y \geq 9 \end{cases} = \begin{cases} 1 & y < 0 \\ \left(\frac{10}{y+16}\right)^4 & 0 \leq y < 9 \\ 0 & y \geq 9 \end{cases}$$

Consequently, we have that

$$\bar{F}_{Y_P}(y) = \begin{cases} 1 & y < 0 \\ \frac{\left(\frac{10}{y+16}\right)^4}{\left(\frac{10}{16}\right)^4} = \left(\frac{16}{y+16}\right)^4 & 0 \leq y < 9 \\ 0 & y \geq 9 \end{cases}$$

The discretized version of each of the $Y_{P,i}$'s is

$$f_{\hat{Y}_P/h}(0) = P\left(\hat{Y}_P = 0\right) = F_{Y_P}\left(\frac{h}{2}\right) = 1 - \bar{F}_{Y_P}\left(\frac{h}{2}\right)$$
$$f_{\hat{Y}_P/h}(k) = P\left(\hat{Y}_P = kh\right) = \bar{F}_{Y_P}\left(\frac{(2k-1)h}{2}\right) - \bar{F}_{Y_P}\left(\frac{(2k+1)h}{2}\right).$$

Now, notice that the probability of a loss occuring is

$$\bar{F}_{Y_L}(0) = \left(\frac{10}{16}\right)^4 = \left(\frac{5}{8}\right)^4 =: \lambda.$$

Therefore, since $N_P = \sum_{i=1}^{N_L} \mathbb{1}_{\{Y_L > 0\}}$, we have

$$G_{N_P}(t) = G_{N_L}(1 - \lambda + \lambda t) = e^{3(1-\lambda+\lambda t-1)} = e^{3\lambda(t-1)},$$

and so $N_P \sim \text{Poi}(3\lambda)$. Using Panjer's Recursion for the $(a, b, 0)$ class (as $N_P \sim \text{Poi}(3\lambda)$) on $\frac{\hat{S}}{h}$, where $\hat{S}$ is the discretized version of $S = \sum_{i=1}^{N_P} Y_{P,i}$, we have

$$g_{\hat{S}/h}(0) = G_{N_P}\left(f_{\hat{Y}_P/h}(0)\right) = e^{3\lambda\left(f_{\hat{Y}_P/h}(0)-1\right)}$$
$$g_{\hat{S}/h}(k) = \sum_{i=1}^{k} \frac{3\lambda i}{k} f_{\hat{Y}_P/h}(i) g_{\hat{S}/h}(k-i).$$

#### 22.1.1.2 *Other Insurance Models based on the Collective Risk Model*

**Definition 52 (Excess-of-loss Insurance)**

*Given a collective risk model $S = \sum_{i=1}^{N} X_i$, if an ordinary deductible $d$ is applied to each risk, then the aggregate loss covered by the insurer is*

$$S^* = \sum_{i=1}^{N} [X_i - d]_+.$$

*Such a contract is known as an **excess-of-loss insurance**.*

**Note**

*By Proposition 25, for an excess-of-loss insurance, we have*

$$E(S^*) = E(N)E[[X-d]_+]$$
$$\mathrm{Var}(S^*) = E(N)\,\mathrm{Var}([X-d]_+) + \mathrm{Var}(N)E[[X-d]_+]^2.$$

**Definition 53 (Stop-loss Insurance)**

*Given a collective risk model $S = \sum_{i=1}^{N} X_i$, a* ***stop-loss insurance*** *is an insurance where an ordinary deductible $d$ is applied on the aggregate loss, i.e.*

$$S^* = [S-d]_+ = \left[\sum_{i=1}^{N} X_i - d\right]_+.$$

**Note**

*By Proposition 14, we have that*

$$E[S^*] = E[[S-d]_+] = \int_d^\infty \bar{F}_S(x)\,dx.$$

**Proposition 30 (A Recursive Formula for the Expected Value of a Stop-loss Insurance)**

*Let $S$ be discrete, taking values in $\mathbb{N}$. Then $E[[S-d]_+]$ can be calculated recursively by*

$$E[[S-(d+1)]_+] = E[[S-d]_+] - \bar{F}_S(d),$$

*for $d = 0, 1, \ldots$, starting from $E(S)$.*

**Proof**

Note that for any $x \in \mathbb{R}_{\geq 0}$, we have $x \in (d, d+1)$ for some $d \in \mathbb{N} \cup \{0\}$. Thus $P(S > x) = P(S > d) = \bar{F}_S(d)$. Following this, we have that

$$\begin{aligned} E[[S-d]_+] &= \int_d^\infty \bar{F}_S(x)\,dx = \int_d^{d+1} \bar{F}_S(x)\,dx + \int_{d+1}^\infty \bar{F}_S(x)\,dx \\ &= \int_d^{d+1} \bar{F}_S(d)\,dx + E[[S-(d+1)]_+] \\ &= E[[S-(d+1)]_+] + \bar{F}_S(d), \end{aligned}$$

as is required. □

---

**Example 22.1.2**

Let $S = \sum_{i=1}^N X_i$, where $N \sim \text{Poi}(4)$, and $X$ has pmf $p_1 = \frac{3}{4}$ and $p_2 = \frac{1}{4}$. Calculate $E[[S-2]_+]$.

**Solution**

Note that we have $E[S] = E[N]E[X] = 4 \cdot \left[\frac{3}{4} + \frac{1}{4}\right] = 5$, and

$$\bar{F}_S(0) = P(S > 0) = P(N > 0) = 1 - e^{-4},$$

where the second equality follows from knowing that $S > 0$ if $N > 0$. Thus

$$E[[S-1]_+] = 5 - 1 + e^{-4} = 4 + e^{-4}.$$

Now

$$\begin{aligned} \bar{F}_S(1) &= P(S > 1) = 1 - P(S=0) - P(S=1) \\ &= 1 - e^{-4} - P(N=1, X=1) \quad {}^{1} \\ &= 1 - e^{-4} - 4e^{-4} \cdot \frac{3}{4} \\ &= 1 - 4e^{-4}. \end{aligned}$$

Thus

$$E[[S-2]_+] = 4 + e^{-4} - 1 + 4e^{-4} = 3 + 5e^{-4}.$$

[1] This follows for $N = 1, X = 1$ is the only possibility for when $S = 1$.

# A Problem Set 1

1. Suppose that the random variable $X$ has density

$$f_X(x) = \frac{\beta(\beta+1)\alpha^\beta x}{(\alpha+x)^{\beta+2}}, \quad x > 0,$$

where $\alpha > 0$ and $\beta > 1$.

(a) Determine the cdf of $X$.

(b) Determine $E[X]$.

(c) Determine the hazard rate function, $h(x)$.

(d) Determine the mean excess loss function $e_X(x)$.

2. Suppose that the loss rv $X$ is an equal mixture of an exponential distribution with mean $\frac{1}{2}$ and a gamma distribution with parameters $\alpha = 3$ and $\theta = \frac{1}{2}$.

(a) Determine the density function of $X$.

(b) Determine the survival function of $X$.

(c) Determine the hazard rate of $X$.

(d) Determine the mean excess loss function.

3. Suppose that $X \mid \Lambda = \lambda \sim \mathrm{Exp}(\lambda)$, and $\Lambda$ has the density function

$$g_\Lambda(\lambda) = \frac{\lambda^{p-1}}{\Gamma(p)\Gamma(1-p)(v-\lambda)^p}, \quad 0 < \lambda < v,$$

where $v > 0$ and $0 < p < 1$. Show that the unconditional distribution of $X$ is $\mathrm{Gam}(p, v)$.

4. A discrete rv $N$ with probability mass function

$$p_n = \frac{1}{n}\left(\frac{\beta}{1+\beta}\right)^n \frac{1}{\ln(1+\beta)}, \quad n \in \mathbb{N} \setminus \{0\},$$

for $\beta > 0$ is said to have a logarithmic distribution.

(a) Find the probability generating function (pgf) $(G(t)$ of $N$. (**Hint**: use the Taylor series $-\ln(10x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$)

(b) Using the same pgf above to determine $E[N]$ and $\text{Var}(N)$. (**Hint**: use Definition 41.)

5. In this course, we often encounter the computation of integrals such as $\int_0^x t^2 e^{-t}\, dt$ and $\int_x^\infty t^2 e^{-t}\, dt$. The following formula can be used to expedite the computation:

$$\int_x^\infty t^n e^{-t}\, dt = \sum_{k=0}^{n} \frac{n!}{k!} x^k e^{-x}, \quad n = 0, 1, \ldots.$$

Also, note that the Gamma function is defined as $\Gamma(\alpha) = \int_0^\infty e^{\alpha-1} e^{-t}\, dt$ for $\alpha > 0$.

(a) Prove the identity above by induction.

(b) Use the identity to show $\Gamma(n+1) = n!$ for $n = 0, 1, \ldots$. Then, show that

$$\int_0^x t^n e^{-t}\, dt = \Gamma(n+1) - \int_x^\infty t^n e^{-t}\, dt = n! - \sum_{k=0}^{n} \frac{n!}{k!} x^k e^{-x}.$$

(c) Consider a loss rv $X \sim \text{Gam}(\alpha, \theta)$ with $\alpha = 3$ and $\theta = 2$, so that it has the pdf

$$f(x) = \frac{\left(\frac{x}{2}\right)^3 e^{-\frac{x}{2}}}{x\Gamma(3)}, \quad x > 0.$$

Compute $\text{Var}(X)$.

# B Problem Set 3

*Problem Set 2 was a bunch of questions from the recommended text.*

1. Suppose that the ground-up loss rv $X$ has the distribution function

$$F_X(x) = 1 - \left(1 - \frac{x}{\theta}\right)^{\alpha}, \quad 0 \le x \le \theta,\ \alpha > 0.$$

   An ordinary deductible $d < \theta$ is applied to each loss.

   (a) Show that the distribution of the amount paid per payment $Y_P$ is the same as the distribution of $X$ with $\theta$ replaced by $\theta - d$.

   (b) Is $X$ an IMRL or DMRL distribution? Justify.

   (c) Determine the loss elimination ratio.

2. The density function of the ground-up loss rv $X$ is given by

$$f_X(x) = \frac{1}{100}\left(q + \frac{1}{100}(1-q)x\right)e^{-\frac{x}{100}}, \quad x > 0,$$

   where $0 < q < 1$. With an ordinary deductible of 100, the expected value of amount paid per payment is known to be 125. Determine the expected value of amount paid per payment if the deductbile level is adjusted to 200.

3. The cdf of a ground-up loss rv $X$ is given by

$$F_X(x) = \frac{2x + x^2}{2b + b^2}, \quad 0 \le x < b.$$

   (a) Does $X$ have a DFR, IFR, or neither? Justify your answer.

   (b) Is $X$ a IMRL distribution, DMRL distribution, or neither? Justify your answer.

   (c) Assume that all losses are subject to a Franchise deductible of $d$ with $d < b$. Find the mean of amount paid per loss.

4. Suppose that the ground-up loss rv $X$ is defined as $X = \frac{1}{Y}$, where $Y$ follows an exponential distribution with the pdf

$$f_Y(t) = 5e^{-5y}, \quad y > 0.$$

Assume that a policy limit of 100, and an ordinary deductible of 10 are applied to all losses. Determine the probability that the amount paid by the insurer on a per payment basis is less than or equal to 50.

5. Assume that the ground-up loss rv $X \sim \text{Unif}(0, 650)$. Suppose there is an ordinary deductible of 150.

   (a) Find the survival function of the amount paid per loss.

   (b) Find the survival function of the amount paid per payment.

   (c) Find the mean excess loss $e_X(150)$.

   (d) Calculate $\text{Var}((X - 150)^2 \mid X > 150)$.

6. let $X$ be the ground-up loss for the current year for an insurer with the following pdf

$$f_X(x) = \frac{375000}{x^4}, \quad x > 50.$$

   (a) Suppose that a Franchise deductible of 100 is applied to the loss. Find $\bar{F}_{Y_L}(y), \bar{F}_{Y_P}(y), E[Y_L]$ and $E[Y_P]$.

   (b) Suppose that an ordinary deductible of 150 is applied to the loss. Calculate the loss elimination ratio for the current year.

   (c) Suppose that an annual inflation rate of 5% will prevail. The insurer would like to model its ground-up loss by $1.05X$ for the next year. To keep the same loss elimination ratio for the next year, what should be the ordinary deductible for $d$ for the next year?

   (d) Suppose that the insurer institutes an ordinary deductible of 100, a coinsurance of 85%, and a **maximum payment** of $1,700$ in the current year. Calculate $\bar{F}_{Y_L}(y), \text{Var}(Y_L)$ and the probability that the amount paid by the insurer on a per payment basis will exceed 50.

7. Total claims for a health plan have a Pareto distribution with $\alpha = 2$ and $\theta = 500$. The health plan implements an increase to physicians

that will pay a bonus of 50% of the amount by which total claims are less than 500; otherwise no bonus is paid. It is anticipated that with the incentive plan, the claim distribution will change to become Pareto with $\alpha = 2$ and $\theta = K$. With the new distribution, it turns out that the expected claims plus the expected bonus is equal to the expected claims prior to the bonus system. Determine the value of $K$.

8. Losses follow a Pareto distribution with $\alpha = 2$ and $\theta = 5000$. An insurance policy pays the following for each loss. There is no insurance payment for the first 1000. For losses between 1000 and 6000, the insurance pays 80%. Losses above 6000 are paid by the insured until the insured has made a total payment of 10000. For any remaining part of the loss, the insurance pays 90%. Determine the expected insurance payment per loss.

# C Problem Set 4

1. Suppose that $N \mid \Lambda = \lambda \sim \text{Poi}(\lambda)$, and $\Lambda \sim \text{Unif}(0, 5)$. Determine the probability that $N \geq 2$.

2. Let $N$ be the number of claims of an insurance portfolio. Assume that $N \mid \Theta = \theta \sim \text{NB}(\theta, 5)$, and $\Theta \sim \text{Unif}(0, 8)$.

   (a) Calculate the expectation of the number of claims.

   (b) Calculate the variance of the number of claims.

   (c) Calculate the probability that there are at least two claims.

3. Assume that $X \mid \Theta = \theta \sim \text{Bernoulli}(\theta)$, and $\Theta \sim \text{Beta}(1, 3, 1)$.

   (a) Show that $X$ follows a Bernoulli distribution and identify its parameter.

   (b) Show that the conditional distribution of $\Theta$, given $X = 0$, is a beta distribution and identify the parameters for this beta distribution.

4. Suppose that $\Lambda \sim \text{Gam}(\alpha, \theta)$ and $N \mid \Lambda = \lambda \sim \text{Poi}(\lambda + \mu)$. We denote the pmf of $N$ by $\{p_k\}_{k \in \mathbb{N} \cup \{0\}}$.

   (a) Show that $N$ has the pgf

   $$G_N(t) = e^{\mu(t-1)}[1 - \theta(t-1)]^{-\alpha}.$$

   (b) Find $p_0$.

   (c) Show that $G_N'(t) = [\mu + \alpha\theta(1 + \theta - \theta t)^{-1}]G_N(t)$, and hence

   $$(1 + \theta)G_N'(t) = \theta t G_N'(t) + [\mu(1 + \theta) + \alpha\theta]G_N(t) - \mu\theta t G_N(t).$$

(d) Use the above identity to show that

$$p_i = \left(\mu + \frac{\alpha\theta}{1+\theta}\right) p_0$$
$$p_{k+1} = \frac{[\mu(1+\theta) + (\alpha+k)\theta]p_k - \mu\theta p_{k-1}}{(1+\theta)(k+1)}, \ k = 1, 2, \ldots.$$

5. Consider a counting rv $N$ with pmf

$$p_0 = 0.2, \ p_1 = 0.3, \ p_2 = 0.3, \ \text{and} \ p_3 = 0.2.$$

Is $N$ is member of the $(a, b, 0)$ class? Justify.

6. Let $N$ be a member of the $(a, b, 0)$ class. Prove that

$$E[N] = \frac{a+b}{1-a}.$$

**Hint**: Show that

$$kp_k = a(k-1)p_{k-1} + (a+b)p_{k-1}, \quad k = 1, 2, \ldots.$$

# D Problem Set 5

1. Let $N$ be a logarithmic rv with pmf

$$p_k = \frac{-1}{k\ln(1-\beta)}\beta^k, \quad k = 1, 2, 3, \ldots,$$

for $0 < \beta < 1$.

(a) Show that the distribution of $N$ is an $(a, b, 1)$ member. Identify the parameters of $a$ and $b$.

(b) Show that the pgf of $N$ is

$$G_N(t) = \frac{\ln(1-\beta t)}{\ln(1-\beta)}.$$

(c) Let $M$ be an rv with pgf

$$G_M(t) = G_N(1-\alpha+at),$$

where $0 < \alpha < 1$. Show that the pgf of $M$ can be expressed as

$$G_M(t) = q + (1-q)\frac{\ln(1-\beta^* t)}{\ln(1-\beta^*)},$$

where

$$q = \frac{\ln(1-\beta(1-\alpha))}{\ln(1-\beta)} \text{ and } \beta^* = \frac{\alpha\beta}{1-\beta(1-\alpha)}.$$

(d) From the above, comment on the distribution of the rv $M$.

2. Let $N$ be the logarithmic rv with pmf

$$p_k = \frac{\beta^k}{k(1+\beta)^k \ln(1+\beta)},$$

and parameter $\beta = 10$. Consider its zero-modified version $N^M$

with $p_0^M = 0.1$.

(a) Find $P\left(N^M > 2\right)$.

(b) Find $E\left[N^M\right]$.

(c) Find $\text{Var}\left(N^M\right)$.

3. Assume that the number of losses $N_L$ have a zero-truncated Poisson rv with parameter $\lambda$.

(a) Identify the pmf and the pgf of $N_L$.

(b) Assume that a loss results in a positive payment with probability $\alpha$ (independently of each other and of the total number of losses). Show that the distribution of the number of positive payments is a zero-modified Poisson rv with parameter $\alpha\lambda$, and identity the probability at 0.

(c) Determine the mean and variance of the number of positive payments.

4. Suppose that $N$ has the following mixed Poisson distribution with $N \mid \Theta = \theta \sim \text{Poi}(\theta)$ and $\Theta$ has the pdf

$$g(\theta) = \frac{1\beta}{e^{-\frac{1}{\beta}(\theta-\lambda)}}, \quad \theta > \lambda,$$

where $\lambda, \beta > 0$.

(a) Prove that

$$P(N = n) = \frac{e^{-\lambda}}{1+\beta} \sum_{k=0}^{n} \frac{1}{(n-k)!} \lambda^{n-k} \left(\frac{\beta}{1+\beta}\right)^k, \quad n = 0, 1, \ldots.$$

(b) Find the moment generating function of $\Theta$.

(c) Comment on the distribution of $N$.

(d) Calculate $E[N]$ and $\text{Var}(N)$.

(e) Consider a ground-up loss rv $X$ which has a lognormal distribution with paramters $\mu = 3.3$ and $\sigma = 2.5$. Suppose a Franchise deductible of 200 is imposed. If the number of losses follows the same distribution as $N$, identify the distribution of the number of (nonzero) payments.

5. Consider the compound rv $S$ with primary distribution $N$ and secondary distribution $M$. The primary distribution $N$ is a zero-

truncated negative binomial rv with $\beta = 1$ and $r = 3$. The pmf of $M$ is given by $f_0 = 0.1$, $f_1 = 0.65$ and $f_2 = 0.25$. Compute the pmf of $S$, i.e. $g_k = P(S = k)$ for $k = 0, 1, 2, 3$.

6. The cdf of a ground-up loss $X$ is given by

$$F_X(x) = 1 - \frac{1458000(45 + 2x)}{(90 + x)^4}, \quad x \geq 0.$$

The insurance policy calls for an ordinary deductible of 30 to be imposed. It is assumed that the number of payments $N_P$ has a logarithmic distribution with $\beta = \frac{7}{4}$.

(a) Determine the expected number of payments.

(b) Find the survival function of the amount paid per payment $Y_P$.

(c) Discretize the severity distribution of $Y_P$ using the method of rounding with a span of $h = 30$. Determine the probability that $P\left(\hat{Y}_P = 90\right)$, where $\hat{Y}_P$ is the discretized version of $Y_P$ with $h = 30$.

(d) Use Panjer's recursion to calculate the discretized distribution of aggregate payments up to a discretized amount paid of 90.

7. The cdf of a ground-up loss is given by

$$F_X(x) = 1 - e^{-\left(\frac{x}{100}\right)^3}, \quad x \geq 0.$$

You are given that $N_L \sim \text{NB}(2, 1.5)$. Suppose that an ordinary deductible of 50 is applied to each individual loss.

(a) Determine the distribution of the number of payments $N_P$.

(b) Find the survival function of the amoubnt paid per payment $Y_P$.

(c) Determine the probability that $P\left(\hat{Y}_P = 120\right)$, where $\hat{Y}_P$ is the discretized version of $Y_P$ with $h = 40$.

(d) Use Panjer's recursion to calculate the discretized distribution of aggregate payments up to a discretized amount paid of 120.

8. Consider the compound rv $S$ with secondary distribution $X \sim \text{Exp}(100)$, and primary distribution $N$ with pmf

| $n$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p_n$ | 0.35 | 0.3 | 0.25 | 0.1 |

(a) Calculate $P(S > 250)$ using the pmf method. You are given that the distribution of $n$-convolution follows Exp(100), i.e.

$$P(X_1 + \ldots + X_n > x) = e^{-\frac{x}{100}} \sum_{j=1}^{n-1} \frac{\left(\frac{x}{100}\right)^j}{j!}.$$

(b) Calculate $P(X > 250)$ using the normal approximation.

# E Additional Material

## E.1 Individual Risk Model: An Alternate View

*This appendix serves to explain why our note of $Z_i = I_i X_i$ is wrong with as mush rigour as we can go for now. There may be hand-wavy parts, but those will be indicated.*

We mentioned, as shown by Klugman, Panjer and Willmot (2012)[1], that for the Individual Risk Model, the aggregate claim is modeled by

$$S = \sum_{i=1}^{n} Z_i$$

where $Z_i$ is a random variable for the potential loss of the $i^{\text{th}}$ insurance policy, while $n$ is fixed. It is claimed that we can also express each $Z_i$ as

$$Z_i = I_i X_i$$

where $I_i$ is an indicator function given by

$$I_i(x) = \begin{cases} 1 & \text{if a claim occurs} \\ 0 & \text{if there are no claims} \end{cases},$$

while $X_i$ is the size of the claim(s) for the $i^{\text{th}}$ policy provided that there is a claim.

[1] Klugman, S. A., Panjer, H. H., and Willmot, G. E. (2012). *Loss Models: From Data to Decisions*. John Wiley & Sons, Inc., 4th edition

One problem that arises is: are $X_i$ and $I_i$ independent? They should be if we wish to define $Z_i$ in such a way. In fact, according to
Klugman et al. in page 177,

> *Let $X_j = I_j B_j$, where $I_1, \ldots, I_n, B_1, \ldots, B_n$ are independent.*

where $X_j$ is our $Z_i$, $I_j$ is our $I_i$, and $B_j$ is our $X_i$.

§ *$Z_i$ is not well-defined* Let us be explicit about the definitions of $I_i$ and $X_i$; we have

$$I_i = \mathbb{1}_{\{Z_i > 0\}}$$
$$X_i = Z_i \mid Z_i > 0$$

However, we observe that such a defintion of $X_i$ is undefined on $Z_i = 0$. So the equation

$$Z_i = I_i X_i$$

is note well-defined.

§ *Independence of $I_i$ and $X_i$* We cannot actually tell if $I_i$ and $X_i$ are independent from each other, as it is equivalent to comparing apples with oranges[2]. Recall from our earlier courses, in particular STAT330, of the following notion:

[2] In fact, I think this analogy fits our case perfectly so.

**Definition (Probability Space)**

*Let $\Omega$ be a sample space, and $\mathcal{F}$ a $\sigma$-algebra defined on $\Omega$[3]. A* ***probability space*** *is the measurable space $(\Omega, \mathcal{F})$ with a probability measure, $f : \mathcal{F} \to [0, 1]$, defined on the space. We denote a probability space as $(\Omega, \mathcal{F}, f)$.*

[3] Note that $(\Omega, \mathcal{F})$ is called a **measurable space**.

As mentioned in an earlier §, $X_i$ is not defined on $Z_i = 0$, while $I_i$ is defined on $Z_i = 0$ [4]. So the sample space for $X_i$ and $I_i$ are not the same, and so their probability measures are not the same as well. Therefore, **it is meaningless to ask if $X_i$ and $I_i$ are independent**.

[4] ✋ This statement is hand-wavy.

Our best attempt at fixing this is probably the following: let

$$Z_i = \sum_{i=1}^{I_i} X_i,$$

which we can then have $X_i$ to be independent from $I_i$. However, interestingly so, this is a similar approach to a Collective Risk Model.

## E.2 *Coherent Risk Measure*

An excerpt from Klugman et al. (2012) [5]:

[5] Klugman, S. A., Panjer, H. H., and Willmot, G. E. (2012). *Loss Models: From Data to Decisions*. John Wiley & Sons, Inc., 4th edition

> *The study of risk measures and their properties has been carried out by authors such as Wang. Specific desirable properties of risk measures were proposed as axioms in connection with risk pricing by Wang, Young, and Panjer and more generally in risk measurement by Artzer et al. The Artzner paper introduced the concept of **coherence** and is considered to be the groundbreaking paper in risk measurement.*

Often, we use the function $\rho(X)$ to denote risk measures. One may think of $\rho(X)$ as **the amount of assets required to protect against adverse outcomes of the risk $X$**.

---

**Definition 54 (Coherent Risk Measure)**

*A **coherent risk measure** is a risk measure $\rho(X)$ that has the following four properties for any two loss rvs $X$ and $Y$:*

1. *(**Subadditivity**)* $\rho(X+Y) \leq \rho(X) + \rho(Y)$.
2. *(**Monotonicity**) If $X \leq Y$ for all possible outcomes, then $\rho(X) \leq \rho(Y)$.*
3. *(**Positive homogeneity**)* $\forall c \in \mathbb{R}_{>0}$, $\rho(cX) = c\rho(X)$.
4. *(**Translation invariance**)* $\forall c \in \mathbb{R}_{>0}$, $\rho(X+c) = \rho(X) + c$

---

*Interpretation of the conditions*

- **Subadditivity**
  - the risk measure (and in return, the capital required to cover for it) for two risks combined will not be greater than for the risks to be treated separately;
  - reflects the fact that there shuld be some diversification benefit from combining risks;
  - this requirement is disputed: e.g. the merger of several small companies into a larger one exposes each of the small companies to the **reputational risks** of the others.
- **Monotonicity**

  - if one risk always has greater losses than the other under all circumstances[6], then the risk measure of the greater risk should always be greater than the other.

- **Positive homogeneity**
  - the risk measure is independent of the currency used to measure it;
  - doubling the exposure to a particular risk requires double the capital, which is sensible as doubling provides no diversification.

- **Translation invariance**
  - there is no additional risk for an additional risk which has no additional uncertainty.

[6] Probabilistically, this means $P(X > Y) = 0$

# F Answers to Problem Sets

** Only final answer is provided for checking (if a final answer exists). For proof problems, hints are provided if a trick is involved.*

## F.1 Problem Set 1

1.(a) $F_X(x) = 1 - \frac{\alpha^\beta(\alpha+(\beta+1)x)}{(\alpha+x)^{\beta+1}}$

(b) $E[X] = \frac{2\alpha}{\beta-1}$

(c) $h(x) = \frac{\beta(\beta+1)x}{(\alpha+x)(\alpha+(\beta+1)x)}$

(d) $e_X(x) = \frac{\alpha+x}{\alpha+(\beta+1)x} \cdot \frac{2\alpha+(\beta+1)x}{\beta-1}$

2.(a) $f_X(x) = (1+2x^2)e^{-2x}$

(b) $\bar{F}_X(x) = (1+x+x^2)e^{-2x}$

(c) $h(x) = \frac{1+2x^2}{1+x+x^2}$

(d) $e_X(x) = \frac{1+x+\frac{x^2}{2}}{1+x+x^2}$

3. When encountering an integral with $e$ to some awkward power, it is usually useful to use substitution to, in a sense, transform that power to something more 'pleasant'.

4.(a) $G(t) = 1 - \frac{\ln(1-\beta(t-1))}{\ln(1+\beta)}$

(b) $E[N] = \frac{\beta}{\ln(1+\beta)}$ ; $\text{Var}(N) = \frac{\beta(1+\beta)}{\ln(1+\beta)} - \left(\frac{\beta}{\ln(1+\beta)}\right)^2$

5.(c) $\text{Var}(X) = \alpha\theta^2 = 12$

## F.2 Problem Set 3

1.(a) $Y_L = [X-d]_+$ ; $\bar{F}_{Y_L}(y) = \left(1 - \frac{d+y}{\theta}\right)^\alpha$ for $0 \le y \le \theta - d$

(b) $X$ is a DMRL distribution.

(c) $\text{LER} = 1 - \left(\frac{\theta-d}{\theta}\right)^{\alpha+1}$

2. $q = \frac{2}{3}$ ; $E[Y_P] = 120$

3.(a) X has an IFR

(b) X is a DMRL distribution

(c) $E[Y_L] = \frac{b^2-d^2+\frac{2}{3}(b^3-d^3)}{2b+b^2}$

4. $P(Y_P \leq 50) = 0.79679$

5.(a) $\bar{F}_{Y_L}(y) = \frac{500-y}{650}$ for $0 \leq y \leq 500$

(b) $\bar{F}_{Y_P}(y) = \frac{500-y}{500}$ for $0 \leq y \leq 500$

(c) $e_X(150) = 250$

(d) $\text{Var}((X-150)^2 \mid X > 150) = 5.5556x10^9$

6.(a)

$$\bar{F}_{Y_L}(y) = \begin{cases} 1 & y < 0 \\ 0.125 & 0 \leq y < 100 \\ 125000y^{-3} & y \geq 100 \end{cases}$$

$$\bar{F}_{Y_P}(y) = \begin{cases} 1 & y < 100 \\ 10^6 y^{-3} & y \geq 100 \end{cases}$$

$$E[Y_L] = 18.75 \quad E[Y_P] = 150$$

(b) $\text{LER} = 0.962963$

(c) maximum payment: $0.85(u-100) = 1700$

$$\bar{F}_{Y_L}(y) = \begin{cases} 1 & y < 0 \\ 125000\left(\frac{y}{0.85} + 100\right)^{-3} & 0 \leq y < 1700 \\ 0 & y \geq 1700 \end{cases}$$

$$\text{Var}(Y_L) = 791.066$$

$$\bar{F}_{Y_P}(50) = 0.2496$$

7. bonus: $Z = \frac{1}{2}(500 - Y)\mathbb{1}_{\{Y<500\}}$ where $Y \sim \text{Pareto}(2, K)$ ; $K = 353.5534$

8. $Y_L = 0.8[X-1000]_+ - 0.8[X-6000]_+ + 0.9[X-u]_+$, where $u$ is the size of the claim such that the insured makes a total payment of 10000, i.e.

$$1000 + 0.2(6000-1000) + (u-6000) = 10000;$$

$E[Y_L] = 2699.362$

## F.3 Problem Set 4

1. $P(N \geq 2) = 0.609433$

2.(a) $E[N] = 20$

(b) $\mathrm{Var}(N) = 260$

(c) $P(N \geq 2) = 0.9375$

3.(a) $X \sim \text{Bernoulli}\left(\frac{1}{4}\right)$

(b) $\Theta \mid X = 0 \sim \text{Beta}(1,4,1)$

4.(b) $p_0 = \frac{e^{-\mu}}{(1+\theta)^\alpha}$

(d) Substitute the power series $G_N(t)$ and $G_N'(t)$ into the given identity, and compare by coefficients.

5. No.

## F.4 Problem Set 5

1.(a) $a = \beta$ ; $b = -\beta$

(b) Use the Taylor expansion $\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$

(d) $M$ is a mixture of a degenerate rv at 0 with weight $q$, and a logarithmic rv with parameter $\beta^*$ and weight $1-q$

2.(a) $P\left(N^M > 2\right) = 0.4037$

(b) $E\left[N^M\right] = 3.7533$

(c) $\mathrm{Var}\left(N^M\right) = 27.199$

3.(a) $p_k^T = \frac{1}{1-e^{-\lambda}} \frac{\lambda^k e^{-\lambda}}{k!}$ ; $G_{N_L}(t) = \frac{e^{\lambda(t-1)} - e^{-\lambda}}{1-e^{-\lambda}}$

(b) $G_{N_P}(t) = \frac{e^{\alpha\lambda(t-1)} - e^{-\lambda}}{1-e^{-\lambda}}$ ; $p_0^M = \frac{e^{-\alpha\lambda} - e^{-\lambda}}{1-e^{-\lambda}}$

(c) $\mathrm{Var}(N_P) = \frac{\alpha\lambda(1-(1+\alpha\lambda)e^{-\lambda})}{(1-e^{-\lambda})^2}$

4.(a) The binomial theorem is useful here.

(b) $M_\Theta(t) = \frac{e^{\lambda t}}{1-\beta t}$

(c) $G_N(t) = e^{\lambda(t-1)} \cdot \frac{1}{1-\beta(t-1)}$ ; $N$ is a product of Poi$(\lambda)$ and Geo$(\beta)$.

(d) $E[N] = \lambda + \beta$ ; $\text{Var}(N) = \lambda + \beta(1+\beta)$

(e) $N_P$ can be viewed as an independent sum of a Poisson rv with parameter $\alpha\lambda$ and a geometric rv with parameter $\alpha\beta$

5. $N$ is an $\left(\frac{1}{2}, 1, 1\right)$ member ; $g_0 = 0.023761$ ;

$$g_k = \frac{\frac{3}{14} f_k + \sum_{j=1}^{k} \left(\frac{1}{2} + \frac{j}{k}\right) f_j g_{k-i}}{0.95}$$

so $g_1 = 0.171006$ ; $g_2 = 0.182776$ ; $g_3 = 0.156716$

6.(a) $E[N_P] = 1.7299$

(b) $\bar{F}_{Y_P} = \frac{13824000(105+2y)}{7(120+y)}$ for $y \geq 0$

(c) $f_0 = 0.197335$ ; $f_1 = 0.28311$ ; $f_2 = 0.17127$ ; $f_3 = 0.10556$

(d) $P(\hat{S} = 90) = 0.10421$

7.(a) $N_P \sim \text{NB}(2, 1.32375)$

(b) $\bar{F}_{Y_P}(y) = \exp\left\{-\left(\frac{y+50}{100}^3\right) + \frac{1}{8}\right\}$ for $y \geq 0$

(c) $P(\hat{Y}_P = 120) = 0.0037584$

(d) $P(\hat{S} = 120) = 0.11926$

8.(a) $P(S > 250) = 0.15083$

(b) $P(S > 250) = 0.4874$

# *Bibliography*


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# *List of Symbols and Abbreviations*

| | |
|---|---|
| crv | continuous random variable |
| DFR | Decreasing Failure Rate |
| DMRL | Decreasing Mean Residual Lifetime |
| drv | discrete random variable |
| $e_X(d)$ | Mean Excess Loss / Mean Residual Lifetime |
| $G_N(t)$ | probability generating function of random variable $N$ |
| $h_X$ | hazard rate of random variable $X$ |
| IFR | Increasing Failure Rate |
| IMRL | Increasing Mean Residual Lifetime |
| LER | Loss Elimination Ratio |
| mgf | moment generating function |
| pf | probability function |
| pdf | probability density function |
| pmf | probability mass function |
| pgf | probability generating function |
| rv | random variable |
| $\bar{F}_X$ | survival function of random variable $X$ |
| $T_L$ | Amount Paid per Loss |
| $T_P$ | Amount Paid Per Payment |
| TVaR | Tail-Value-at-Risk |
| VaR | Value-at-Risk |

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